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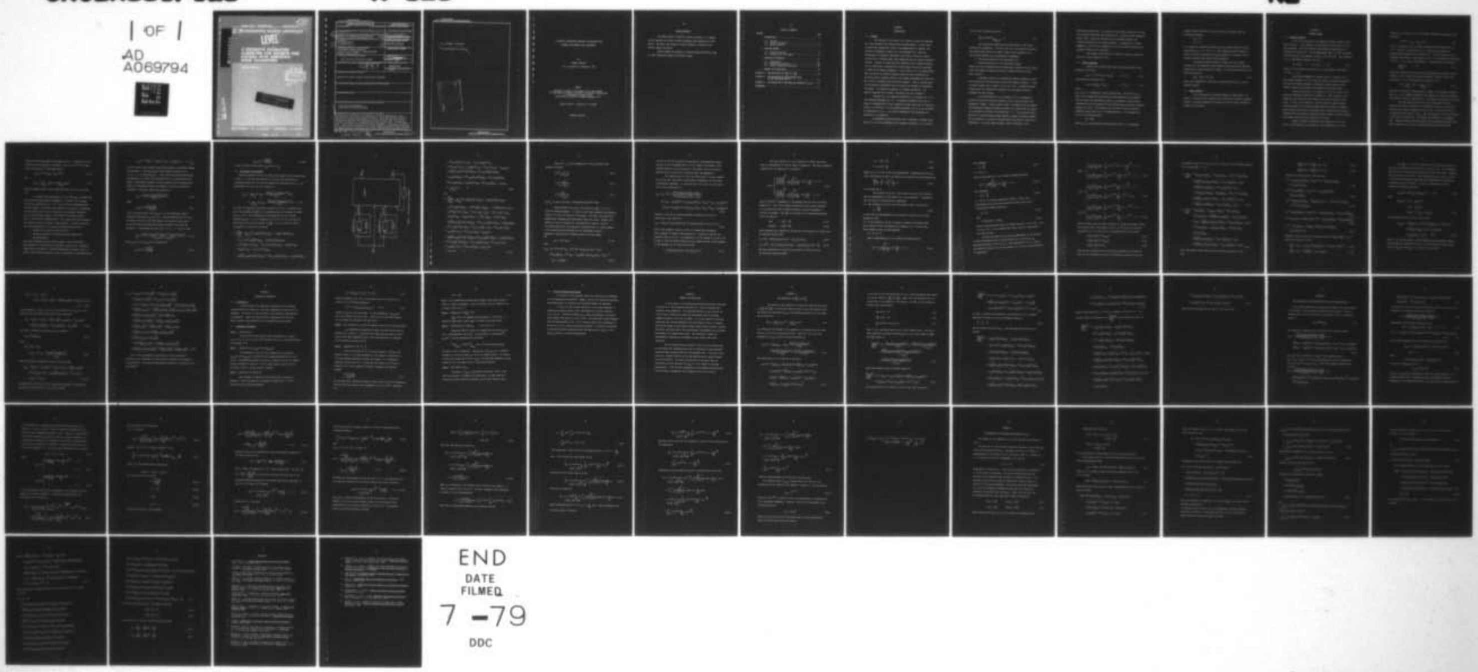

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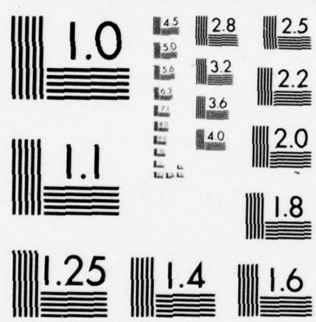

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A RECURSIVE ESTIMATION ALGORITHM FOR DISCRETE-TIME SYSTEMS WITH UNKNOWN NOISE PARAMETERS

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The purpose of this report is to develop a recursive algorithm for state estimation for systems with uncertain models. The algorithm uses a combined detection-estimation scheme whereby the set in which the uncertainties are contained is detected, and appropriate estimator is then used. The approach used is an extension of a weighted minimax performance criteria to the dynamic case. Since global optimal solution is not possible, an approximate algorithm is derived which only optimizes the stage-by-stage performance without changing any past decisions. The expressions for the algorithm and an approximation of

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20. ABSTRACT (continued)

its performance are derived.

A RECURSIVE ESTIMATION ALGORITHM FOR DISCRETE-TIME SYSTEMS WITH UNKNOWN NOISE PARAMETERS

BY

NATHAN GUEDALIA

B.S., University of Tennessee, 1976

THESIS

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Thesis Advisor: Professor A. H. Haddad

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CHAPTER 1

INTRODUCTION

1.1. General

The purpose of this thesis is to develop a recursive algorithm for state estimation for systems with uncertain models. In many areas of electrical engineering, control and communication for example, the problem of estimating a signal is a very important one. When the system model is completely specified it was shown [1] that an optimal solution can be obtained under some assumptions and for various optimality criteria. However, in many practical situations such a complete description is unavailable, therefore the system's model can only be specified in terms of the unknown parameters. It was shown in [2], [3] among others that the use of an incorrect system's model and noise covariances in particular have a dramatic effect on the system performance and often will give rise to a problem called "Filter Divergence". When a complete model of the system is unavailable there are mainly two methods for estimating its states: (a) Adaptive Estimation, (b) Mimimax Estimation. The adaptive approach can be roughly classified into four categories.

Bayesian [4], [5], Maximum Likelihood [6]; Correlation [6], [7]; and Covariance Matching [6], [7]. In Bayesian Estimation one tries to obtain a representation of the a posteriori density function viz $f(X_i,\theta|Y_i)$ where X_i is the state, θ is the unknown parameter, and Y_i is the collection of observations up to i. An a priori knowledge of the system and its statistics is a necessity.

The Maximum Likelihood method tries to estimate a parameter when there is no a priori knowledge of the parameter available, i.e. one tries

to solve the "Likelihood Equation".

$$\frac{\partial}{\partial\theta}\,[\ln f(Y_i|\theta)]\Big|_{\theta=\hat{\theta}_{ML}(Y_i)} = 0. \tag{1.1}$$

The correlation method had been used mainly in time series analysis and econometrics. The basic idea is to correlate the output of the system either directly or after a known linear operation on it. Two different methods can be developed by considering

a. The autocorrelation of the observation $\{y_i\}$.

b. The autocorrelation of the innovation process.

Usually the above schemes have good large sample properties but for small sample the adaptive schemes yield state estimates which have large error variance.

A minimax solution to an estimation problem, minimizes a given cost function for the worst case values of the unknown parameter. A more in depth treatment can be found in [1], [10], [11]. In general Minimax schemes yield too conservative estimates for large sample and good estimates for small sample.

A third method which can be thought of as an intermediate between classical and adaptive estimation is the "Joint Detection and Estimation" schemes. Several authors ([12], for example) have considered joint detection-estimation schemes in which it is assumed that the signal to be estimated is present with probability $P < 1$. The system structure consist of a detector which decides whether a signal is actually present, and an estimator to provide the estimate when the detector's decision is affirmative. The above scheme assumes a priori knowledge of the

underlying probabilities. An extension of the above approach to the case of M possible modes have been developed [13]. Another approach suggested Pearson [18]: design the estimator $\hat{X}_i$ as a Minimax estimator and perform the detection via classical methods. An extension to [8] was suggested by [14], [15] by assigning a cost to the entire detection and estimation scheme and to minimize it.

The purpose of this work is to develop a recursive algorithm that will be the extension to [14].

1.2. Problem Statement

Consider the discrete-time linear system satisfying a state equation of the form:

$$X_{k+1} = \phi\,(k+1,k)\,X_k + \Gamma_{0k}\,\bar{w}_k, \qquad k = 0,1,... \tag{1.2}$$

and observed in additive noise

$$y_{k+1} = C_{k+1}X_{k+1} + v_{k+1}, \qquad k = 0,1,... \tag{1.3}$$

where

X_k, y_k are n, m dimensional vectors respectively. The vectors $\bar{w}_k$, v_k, and p, and m dimensional uncorrelated, white Gaussian noise processes with zero mean and covariances Q_k and R_k, respectively, where without of generality we can consider only the problem where the unknown parameter appears only in Q_k. The extension for the case in which the unknown parameter appears also in R_k is straight-forward. It is assumed that Q_k is of the following form:

$$Q_k = \theta Q_{0k}$$

where Q_{0k} is a known positive definite matrix, and θ - is an unknown

parameter which satisfies at least one of the two bounds under the following hypotheses:

$$H_0: \theta \le b$$

$$H_1: \theta \le a$$

It is desired to estimate the state X_k of the system based on the past observations $\{y_i, 1 \le i \le k\} \equiv Y_k$. The proposed approach attempts to derive a recursive algorithm for state estimation in uncertain linear dynamical systems with unknown noise parameters.

The proposed scheme attempts to retain the small sample properties of the standard minimax estimator while having good adaptive properties for large observation records. For convenience equation (1.2) may be rewritten as

$$X_{k+1} = A_k X_k + \sqrt{\theta}\,\Gamma_k w_k \tag{1.4}$$

where w_k is normalized to have unit covariance, and hence Γ_k is appropriately defined.

1.3. Thesis Outline

In the following the proposed approach is investigated. In Chapter 2 we describe the approach used to derive the recursive algorithm. Chapter 3 discusses computation procedure and some relevant computation aspects of the problem.

CHAPTER 2

PROPOSED SCHEME

2.1. Proposed Approach

Since our uncertainty on the unknown parameters appears in the form of bounds, a joint detection and estimation scheme seems to be the most promising. The joint detection and estimation approach can be described as follows [14]. Let the measurement space $Y_k \equiv R^{mk}$ be divided into two exclusive and exhaustive regions $Y_1(k)$ and $Y_2(k)$. The estimation is to be performed according to the rule

$$\hat{X}_k = \hat{X}_k(i) \quad \text{if} \quad Y_k \in Y_i(k), \; i = 0,1 \tag{2.1}$$

where $\hat{X}_k$ is the estimate of X_k and $\hat{X}_k(i)$ is the estimator appropriate to $\theta \in H_i$, $i = 0,1$.

One approach suggested by Pearson [8] is to design $\hat{X}_i$ to be the minimax estimator of x for $\theta \in H_i$ with respect to some cost function, and then given these estimators to design an optimal detector using classical detection techniques like Generalized Likelihood Ratio.

Another possibility is to assign a cost to the overall joint detection estimation scheme and to optimize it. This approach is very appealing since detection and estimation are closely related to each other. In fact it was shown [9], [10] that both the causal minimum variance estimate of an arbitrary signal process corrupted by additive white Caussian noise and its associated error covariance matrix can be obtained from the sequential Likelihood Ratio by means of simple formulas. This refined approach can be described as follows:

Assign an overall conditional cost denoted by $C(\theta)$ to the overall joint detection and estimation. One possibility for a cost

function is to choose $C(\theta)$ to be the CMSE "conditional mean square error" defined by

$$C(\theta) \triangleq E\{ \| X_{k+1} - \hat{X}_{k+1} \|^2 | \theta\} . \quad (2.2)$$

Since θ is unknown in (2.2) we cannot proceed with the minimization of (2.2). Instead a modification to (2.2) which can be thought as a minimax related approach was introduced [14]. The cost function will be optimized with respect to the decision rule (i.e. the estimators appropriate to $\theta \in H_i$, i = 0,1; and to the detector). We require that the estimator

$$\max_{\theta \in H_i} C(\theta) \quad , \quad i = 0,1 \quad (2.3)$$

Since this problem is a multiple objective optimization problem, a direct solution of (2.2) cannot be further considered. Instead a scalar cost functional $\tilde{C}$ is to be minimized and is defined by (2.4)

$$\tilde{C} = \lambda_0 \max_{\theta \in H_1} C(\theta) + \lambda_1 \max_{\theta \in H_0} C(\theta) \quad (2.4)$$

which can be viewed as a generalization of Magill's performance criterion. Here λ_0, λ_1 are weighting parameters that measure the relative importance associated to errors in each region H_i. The analogy can be drawn as follows:

The case where the regions in our problem formulation are mutually exclusive and exhaustive is equivalent to Magill's, where the λ_i represent the a priori probability that $\theta_i \in H_i$. The objective now is to minimize $\tilde{C}$ with respect to $\hat{X}_{k+1}$. The cost functional (2.4) was proposed in [14] for the static case. We consider its extension to the dynamic case. However, both the decision regions and the estimators depend on the entire observation record Y_k, and requires increasing computational

complexity and growing memory requirements with k. Consequently, a sub-optimal recursive approach is proposed. Let (2.2) and (2.4) at stage k+1 be written with a slight modification.

$$C_{k+1}(\theta) = E\{\| X_{k+1} - \hat{X}_{k+1} \|^2 | \theta\} \tag{2.5}$$

and

$$\tilde{C}_{k+1} = \max_{\theta \in H_1} C_{k+1}(\theta) + \lambda \max_{\theta \in H_0} C_{k+1}(\theta) \tag{2.6}$$

where we assumed without loss of generality that λ_0 in (2.4) is equal to 1.

In minimizing the expression (2.6) at stage X_{k+1} we assume that the earlier decisions remains unchanged i.e. we don't try to update the earlier estimates $\hat{X}_j(i)$, $j \leq k$, $i=0,1$, and the corresponding past decision regions. We restrict the problem to be recursive by searching for an estimate which is only a function of the present observation y_{k+1} and the previous outputs of the states of the states such as $\hat{X}_k(i)$, $i=0,1$ and the likelihood ratio. As a result the overall scheme is not globally optimal but stage by stage optimal. In order to continue with the minimization of (2.6) several approaches are possible.

1. Proceed with an exact solution of (2.6)
2. Assume an estimator's structure and then proceed with the minimization.

The second approach was chosen mainly because of the possibility that approach (1) can lead to an intraceable solution [15]. Therefore, assume that the estimator's structure appropriate to region $\theta \in H_i$ is given by the following expression, which is motivated by the Kalman filter:

$$\hat{X}_{k+1}(i) = A_k\hat{X}_k(i) + K_{k+1}(i)\,(y_{k+1} - C_{k+1}A_k\hat{X}_k(i)) \qquad i = 0,1 \tag{2.7}$$

We still haven't said anything about the functional relationship between the estimate of X denoted $\hat{X}_{k+1}$ and the estimator appropriate for $\theta \in H_i$, i = 0,1 denoted $\hat{X}_{k+1}(i)$. Since we have shown already that the introduction of a scalar cost functional (2.6) is a generalization of Magill's performance criterion, we have decided to use the generalized Magill's relationship between the estimate of X and the appropriate estimators. Therefore, choose the estimate by

$$\hat{X}_{k+1} = \frac{\hat{X}_{k+1}(0) + \Lambda_{k+1}(1,0)\,\hat{X}_{k+1}(1)}{1 + \Lambda_{k+1}(1,0)} \tag{2.8}$$

where

$$\Lambda_{k+1}(1,0) \triangleq \frac{f(Y_{k+1}|\theta_i)}{f(Y_{k+1}|\theta_0)} \tag{2.9}$$

is the likelihood-Ratio, and $f(Y_{k+1}|\theta_i)$ is the conditional density function of the output Y_{k+1} given θ_i. From the definition of (2.10) it is apparent that the avove relationship can be used for cases where we have more than two associated regions, therefore, more than two estimators. Illustrating the point for M = 3 i.e. i = 0,1,2 we have

$$\hat{X}_{k+1} = \frac{\hat{X}_{k+1}(0) + \Lambda_{k+1}(1,0)\,\hat{X}_{k+1}(1) + \Lambda_{k+1}(2,0)\,\hat{X}_{k+1}(2)}{1 + \Lambda_{k+1}(1,0) + \Lambda_{k+1}(2,0)} \tag{2.10}$$

where $\Lambda_{k+1}(1,0)$ and $\Lambda_{k+1}(2,0)$ are defined as:

$$\Lambda_{k+1}(1,0) \triangleq \frac{f(Y|\theta_1)}{f(Y|\theta_0)} \tag{2.11A}$$

$$\Lambda_{k+1}(2,0) = \frac{f(Y|\theta_2)}{f(Y|\theta_0)} \qquad (2.11B)$$

In what follows we shall denote $\Lambda_{k+1}(1,0)$ by Λ_{k+1}.

2.2. Derivation of the Scheme

Based on equation (2.8) we can draw a block diagram of the system given in Fig. 1. In our case the detector is a device that calculates the relationship given by (2.8) and also updates the likelihood ratio Λ_k. The substitution of (2.8) into (2.6) results in:

$$\tilde{C}_{k+1} = \max_{\theta \in H_1} E\left\{\left\| X_{k+1} - \frac{(\hat{X}_{k+1}(0) + \Lambda_{k+1}\hat{X}_{k+1}(1))}{1+\Lambda_{k+1}}\right\|^2 \Big| \theta\right\}$$

$$+ \lambda \max_{\theta \in H_0} E\left\{\left\| X_{k+1} - \frac{(\hat{X}_{k+1}(0) + \Lambda_{k+1}\hat{X}_{k+1}(1))}{1+\Lambda_{k+1}}\right\|^2 \Big| \theta\right\} \qquad (2.12)$$

Utilizing the estimator's structure as given by (2.7) it is apparent that we can choose $K_{k+1}^{(0)}$ and $K_{k+1}^{(1)}$ as the parameters that will minimize the CMSE $\tilde{C}_{k+1}$ in (2.12). We now substitute the expressions for the filtered estimates $\hat{X}_{k+1}(i)$, $i = 0,1$ as defined by (2.7) into (2.14), and take the derivatives of (2.12) with respect to $K_{k+1}^{(i)}$, $i = 0,1$ as shown in detail in Appendix A to arrive at the result.

$$0 = \frac{\partial \tilde{C}_{k+1}}{\partial K_{k+1}^{(0)}} = \max_{\theta \in H_1} \Big\{E\Big\{-\eta_0 A_k \tilde{X}_k(0)\tilde{X}_k'(0)A_k'C_{k+1}' - \eta_1 A_k \tilde{X}_k(1)\tilde{X}_k'(0)A_k'C_{k+1}'$$

$$- (\eta_0 + \eta_1)\sqrt{\theta}\,\Gamma_k w_k \tilde{X}_k'(0)A_k'C_{k+1}' - \eta_0\sqrt{\theta}\, A_k \tilde{X}_k(0) w_k' \Gamma_k' C_{k+1}'$$

$$- \eta_1\sqrt{\theta}\, A_k \tilde{X}_k(1) w_k' \Gamma_k' C_{k+1}' - (\eta_0 + \eta_1)\theta\,\Gamma_k w_k w_k' \Gamma_k' C_{k+1}' - \eta_0 A_k \tilde{X}_k(0) v_{k+1}'$$

$$- \eta_1 A_k \tilde{X}_k(1) v_{k+1}' - (\eta_0 + \eta_1)\sqrt{\theta}\,\Gamma_k w_k v_{k+1}'$$

$$+ \eta_0 K_{k+1}^{(0)}[C_{k+1}A_k\tilde{X}_k(0)X_k'(0)A_k'C_{k+1}' + \sqrt{\theta}\, C_{k+1}\Gamma_k w_k \tilde{X}_k'(0)A_k'C_{k+1}' + v_{k+1}\tilde{X}_k'(0)A_k'C_{k+1}'$$

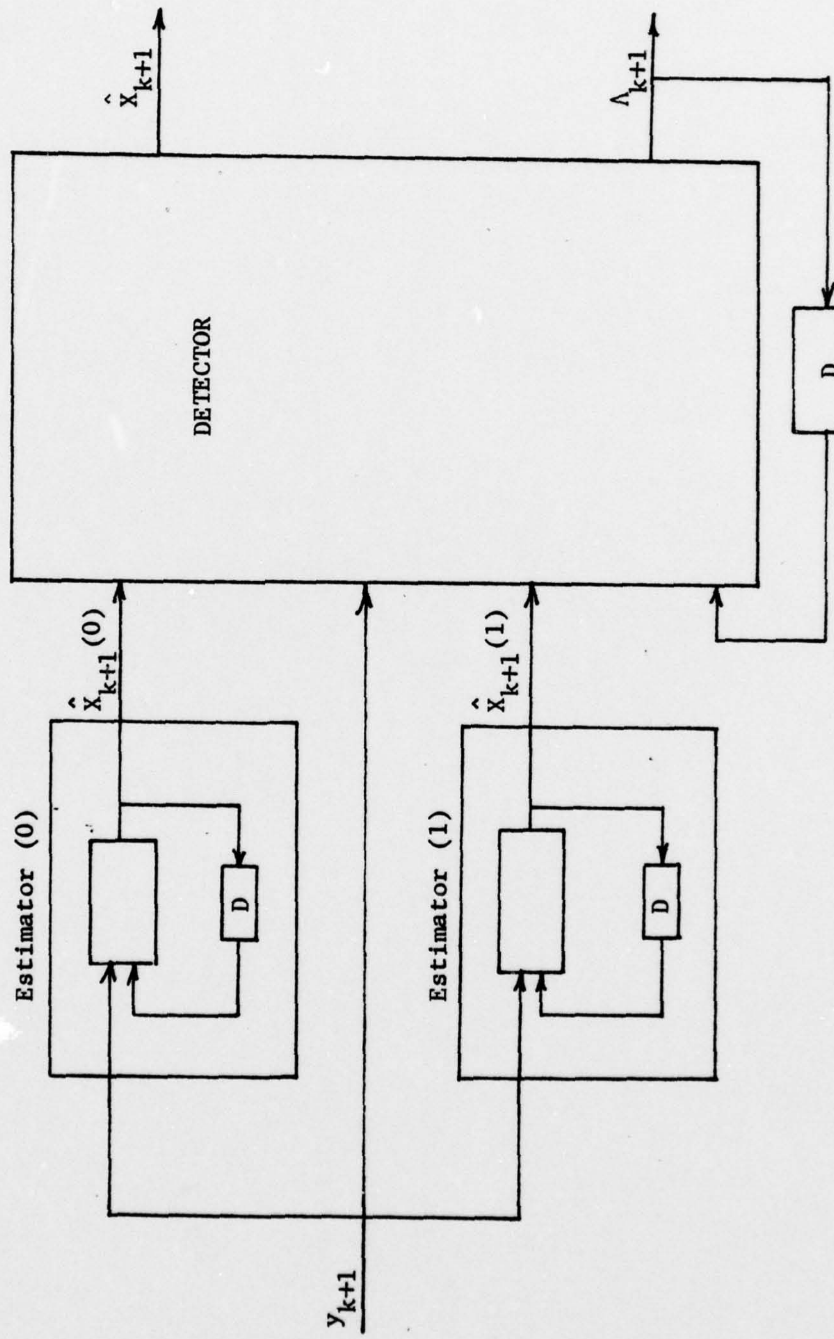

Figure 1. The system's block diagram.

$$+ \sqrt{\theta}\, C_{k+1} A_k \tilde{X}_k(0) w_k' \Gamma_k C_{k+1}' + \theta C_{k+1} \Gamma_{0k} w_k w_k' \Gamma_k' C_{k+1}'$$

$$+ \sqrt{\theta}\, v_{k+1} w_k' \Gamma_k' C_{k+1}' + C_{k+1} A_k \tilde{X}_k(0) v_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k v_{k+1}' + v_{k+1} v_{k+1}']$$

$$+ \eta_1 K_{k+1}^{(1)} [C_{k+1} A_k \tilde{X}_k(1) \tilde{X}_k'(1) A_k' C_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k \tilde{X}_k'(0) A_k' C_{k+1}'$$

$$+ v_{k+1} \tilde{X}_k'(0) A_k' C_{k+1}' + \sqrt{\theta}\, C_{k+1} A_k \tilde{X}_k(1) w_k' \Gamma_k' C_{k+1}' + \theta C_{k+1} \Gamma_k w_k w_k' \Gamma_k' C_{k+1}'$$

$$+ \sqrt{\theta}\, v_{k+1} w_k' \Gamma_k' C_{k+1}' + C_{k+1} A_k \tilde{X}_k(1) v_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k v_{k+1}' + v_{k+1} v_{k+1}'$$

$$+ \lambda \max_{\theta \in H_0} \{E[\ldots \qquad]\} \tag{2.13}$$

and

$$0 = \frac{\partial \tilde{C}_{k+1}}{\partial K_{k+1}^{(1)}} = \max_{\theta \in H_0} \Big\{ E\Big\{ -\eta_1 A_k \tilde{X}_k(0) \tilde{X}_k'(1) A_k' C_{k+1}' - \eta_2 A_k \tilde{X}_k(1) \tilde{X}_k'(1) A_k' C_{k+1}'$$

$$- \eta_1 \sqrt{\theta}\, \Gamma_k w_k \tilde{X}_k'(1) A_k' C_{k+1}' - \eta_2 \sqrt{\theta}\, \Gamma_k w_k \tilde{X}_k'(1) A_k' C_{k+1}' - \eta_1 \sqrt{\theta}\, A_k \tilde{X}_k(1) w_k' \Gamma'_k C_{k+1}'$$

$$- \eta_2 \sqrt{\theta}\, A_k \tilde{X}_k(1) w_k' \Gamma'_k C_{k+1}' - \eta_1 \theta \Gamma_k w_k w_k' \Gamma'_k C_{k+1}' - \eta_2 \theta \Gamma_k w_k w_k' \Gamma'_k C_{k+1}'$$

$$- \eta_1 A_k \tilde{X}_k(0) v_{k+1}' - \eta_2 A_k \tilde{X}_k(1) v_{k+1}' - \eta_1 \sqrt{\theta}\, \Gamma_k w_k v_{k+1}' - \eta_2 \sqrt{\theta}\, \Gamma_k w_k v_{k+1}'$$

$$\eta_1 K_{k+1}^{(0)} [C_{k+1} A_k \tilde{X}_k(0) X_k'(1) A_k' C_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k \tilde{X}_k'(1) A_k' C_{k+1}' + v_{k+1} \tilde{X}_{k+1}' A_k' C_{k+1}'$$

$$+ \sqrt{\theta}\, C_{k+1} A_k \tilde{X}_k(0) w_k' \Gamma'_k C_{k+1}' + \theta C_{k+1} \Gamma_k w_k w_k' \Gamma'_k C_{k+1}' + \sqrt{\theta}\, v_{k+1} w_k' \Gamma'_k C_{k+1}'$$

$$+ C_{k-1} A_k \tilde{X}_k(0) v_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k v_{k+1}' + v_{k+1} v_{k+1}']$$

$$+ \eta_0 K_{k+1}^{(1)} [C_{k+1} A_k \tilde{X}_k(1) \tilde{X}_k'(1) A_k' C_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k \tilde{X}_k'(1) A_k' C_{k+1}' + v_{k+1} \tilde{X}_k'(1) A_k' C_{k+1}'$$

$$+ \sqrt{\theta}\, C_{k+1} A_k \tilde{X}_k(1) w_k' \Gamma_k' C_{k+1}' + \theta C_{k+1} \Gamma_k w_k w_k' \Gamma_k' C_{k+1}' + \sqrt{\theta}\, v_{k+1} w_k' \Gamma_k' C_{k+1}'$$

$$+ C_{k+1} A_k \tilde{X}_k(1) v_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k v_{k+1}' + v_{k+1} v_{k+1}'] \Big\} \Big\}$$

$$+ \lambda \max \{E[\cdots \qquad]\} \tag{2.14}$$

Where η_0, η_1, η_2 are combinations of the likelihood ratio defined as follows:

$$\eta_0 \triangleq \frac{1}{(1+\Lambda_{k+1})^2} \tag{2.15}$$

$$\eta_1 \triangleq \frac{\Lambda_{k+1}}{(1+\Lambda_{k+1})^2} \tag{2.16}$$

$$\eta_2 \triangleq \frac{\Lambda_{k+1}^2}{(1+\Lambda_{k+1})^2} \tag{2.17}$$

and $\tilde{\eta}_0$, $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are their corresponding expected values.

These equations will have to be solved for $K_{k+1}^{(i)}$, i = 0,1, to yield the desired estimate. In order to solve these equations the expected values in (2.13)-(2.14) have to be evaluated. Note that only conditional expectations, assuming past observations fixed, are taken. However, the question of how to represent the likelihood-ratio arises? Since we have shown before that the sequential likelihood ratio is closely related to the minimum variance estimate and its associated variance, a sequential likelihood ratio formula given by [10] was chosen.
Let

$$\Lambda_{k+1} = \exp\,(L_{k+1}) \tag{2.18}$$

then

$$\begin{aligned} L_{k+1} = L_k &+ \tfrac{1}{2}\ln\det R_{k+1} - \tfrac{1}{2}\ln\det\,(C_{k+1}P_{k+1|k}C'_{k+1} + R_{k+1}) \\ &+ \tfrac{1}{2}\,y'_{k+1}R^{-1}_{k+1}y_{k+1} - \tfrac{1}{2}\,(y_{k+1} - C_{k+1}A_k\hat{X}_k)'(C_{k+1}P_{k+1|k}C'_{k+1} + R_{k+1})^{-1}\cdot \\ &(y_{k+1} - C_{k+1}A_k\hat{X}_k) \end{aligned} \tag{2.19}$$

In order to find the appropriate expectations, the probability density function of the incremented part in (2.19) needs to be derived. For a complete analysis of the derivation of the density and the resulting expected values involving the likelihood-ratio see Appendix B.

The formula given in (2.19) has three parts, the first depends only on the past, the second is measurement independent, and the third is measurement dependent. It was shown (B.3)-(B.6) that the likelihood ratio can be represented as:

$$L_{k+1} = L_k + \frac{1}{2} \ln \left[\frac{\det(C_{k+1}P_{k+1|k}(0)C'_{k+1} + R_{k+1})}{\det(C_{k+1}P_{k+1|k}(1)C'_{k+1} + R_{k+1})} \right]$$
$$+ \frac{1}{2} (y_{k+1} - C_{k+1}A_k\hat{X}_k(0))'(C_{k+1}P_{k+1|k}(0)C'_{k+1} + R_{k+1})^{-1}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))$$
$$- \frac{1}{2} (y_{k+1} - C_{k+1}A_k\hat{X}_k(1))'(C_{k+1}P_{k+1|k}(1)C'_{k+1} + R_{k+1})^{-1}(y_{k+1} - C_{k+1}A_k\hat{X}_k(1)) \qquad (2.20)$$

Equation (2.20) may be simply expressed as defined in (B.1)-(B.7) which with (2.18) results in:

$$\Lambda_{k+1} = \exp\{\gamma_k + \frac{1}{2} X'_k(0)G_k(0)X_k(0) - \frac{1}{2} X'_k(1)G_k(1)X_k(1)\} \equiv \exp(\gamma_k + Z) \qquad (2.21)$$

Each of the quadratic forms in (2.20) is a generalized chi-square with n degrees of freedom, (see Appendix B). In order to evaluate the expected value of (2.21), or a combination of it as appears in (2.13)-(2.14) we need to find the density function of the random variable Z which appears in the exponent of (2.21) and has the form:

$$Z = \frac{1}{2} [X'_k(0)G_k(0)X_k(0) - X'_k(1)G_k(1)X_k(1)] \qquad (2.22)$$

The exact density of Z is not possible to express explicitly hence an approximation is used as shown in Appendix B. The final asymptotic expression for the density of Z is given by

$$f_z(t) \cong \begin{cases} \left[\dfrac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right]^{\frac{n}{2}} \dfrac{t^{\frac{n}{2}-1}}{2^n(\sigma_1\sigma_2)^n\Gamma(\frac{n}{2})} \exp\left\{-\dfrac{t}{2\sigma_1^2}\right\}, & t > 0 \\ \left[\dfrac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right]^{\frac{n}{2}} \dfrac{(-t)^{\frac{n}{2}-1}}{2^n(\sigma_1\sigma_2)^n\Gamma(\frac{n}{2})} \exp\left\{\dfrac{t}{2\sigma_2^2}\right\}, & t < 0 \end{cases} \tag{2.23}$$

where σ_1^2 and σ_2^2 are parameters to be obtained from the first and second moments of Z. Due to the lengthy derivation only the general approach will be given here, and a detailed analysis is given in Appendix C. As shown in Appendix C the mean and variance of the approximated density (2.23) are given by

$$\text{mean} = n(\sigma_1^2 - \sigma_2^2) \tag{2.24}$$

$$\text{variance} = 2n(\sigma_1^4 + \sigma_2^4) \tag{2.25}$$

These parameters are compared to the mean β_0 and variance β_1 of Z which can be computed directly from

$$\beta_0 = E\{Z\} = \frac{1}{2} E\{X_k'(0)G_k(0)X_k(0) - X_k'(1)G_k(1)X_k(1)\} \tag{2.26}$$

$$\beta_1 = E\{Z^2\} - \beta_0^2 = \frac{1}{2} E\{[X_k'(0)G_k(0)X_k(1) - X_k'(1)G_k(1)X_k(1)]^2\} - \beta_0^2 \tag{2.27}$$

The exact expressions of β_0 and β_1 are given by (C.30) and (C-31). The resulting equations become

$$\beta_0 = n(\sigma_1^2 - \sigma_2^2) \tag{2.28}$$

$$\beta_1 = 2n(\sigma_1^4 + \sigma_2^4) \tag{2.29}$$

where β_0, β_1 in (2.28)-(2.29) are assumed known. Substituting (2.28) into (2.29) and solving for $\sigma_2^2 \equiv \xi$ we arrive at an equation of the following form,

$$\left[\left(\frac{\beta_0}{n} + \xi\right)^4 + \xi^4\right] - \frac{\beta_1}{4n^2} = 0 \tag{2.30}$$

to be solved for ξ.

The solution for σ_2^2 as in (2.30) may not be real which implies that our approximation to the density of Z is not adequate. Consequently the following two alternatives are considered:

(a) The expression for σ_2^2 is real and positive, which is satisfied if

$$\beta_1 > \frac{2\beta_0^2}{n} \tag{2.31}$$

In this case the approximation for the density is given by (2.23) with parameters σ_1^2, σ_2^2.

(b) If the expression for σ_2^2 is complex or negative the general model (2.23) cannot be used to approximate the density of Z. In this case, three different cases are distinguished

(1) If $\beta_0 > 0$

then we approximate $f_Z(\cdot)$ by a single Chi-Squared density

$$f_Z(t) = \frac{t^{\frac{n}{2}-1}}{2^{n/2}\sigma_1^n \Gamma(\frac{n}{2})} \exp\left\{\frac{-t}{2\sigma_1^2}\right\}, \ t > 0 \tag{2.32}$$

with parameter

$$\sigma_1^2 = \frac{\beta_0}{n} \tag{2.33}$$

(2) If $\beta_0 < 0$

then we approximate $f_Z(\cdot)$ by a single Chi-Squared density

$$f_Z(t) = \frac{(-t)^{\frac{n}{2}-1}}{2^{n/2}\sigma_1^n \Gamma(\frac{n}{2})} \exp\left\{\frac{t}{2\sigma_2^2}\right\}, \; t < 0 \tag{2.34}$$

with parameter

$$\sigma_2^2 = -\frac{\beta_0}{n} \tag{2.35}$$

(3) If $\beta_0 = 0$ we shall assume that $P(Z=0) = 1$ Note; that the motivation for this case stems from the fact that at the initialization stage we may have

$$\hat{X}_0 = \hat{X}_1 \tag{2.36}$$

then

$$Z = \frac{1}{2}\hat{X}_0(G(0) - G(1)\hat{X}_0 \tag{2.37}$$

which in many practical cases will either have $G(0) > G(1)$, $G(0) < G(1)$, or $G(0) = G(1)$. It is unlikely that such a case will occur after the initial stage.

Now that the density $f_Z(t)$ has been determined, we now proceed to find the expected value of the functions η_0, η_1, and η_2 of Λ_{k+1} previously defined by (2.15)-(2.17). The expectations of η_0, η_1, and η_2 are derived by using (2.21) and the density of Z (2.23), in the definition of expectation.

$$E\{\eta_0\} = \begin{cases} \dfrac{1}{[2(\sigma_1^2+\sigma_2^2)]^{n/2}} \displaystyle\sum_{n=0}^{\infty} n(-e^{-\gamma})^{n-1}\left(1+n+\frac{1}{2\sigma_1^2}\right)^{-\frac{n}{2}}, & t>0 \\ \dfrac{1}{[2(\sigma_1^2+\sigma_2^2)]^{n/2}} \displaystyle\sum_{n=0}^{\infty} n(-e^{\gamma})^{n-1}\left(\frac{1}{2\sigma_2^2}+n-1\right)^{-\frac{n}{2}}, & t<0 \end{cases} \tag{2.38}$$

$$E\{\eta_1\} = \begin{cases} \dfrac{e^{-\gamma}}{[2(\sigma_1^2+\sigma_2^2)]^{n/2}} \displaystyle\sum_{n=0}^{\infty} n(-e^{\gamma})^{n-1}\left(\frac{1}{2\sigma_1^2}+n-2\right)^{-\frac{n}{2}}, & t>0 \\ \dfrac{e^{\gamma}}{[2(\sigma_1^2+\sigma_2^2)]^{n/2}} \displaystyle\sum_{n=0}^{\infty} n(-e^{\gamma})^{n-1}\left(\frac{1}{2\sigma_2^2}+n\right)^{-\frac{n}{2}}, & t<0 \end{cases} \tag{2.39}$$

$$E\{\eta_2\} = \begin{cases} \dfrac{1}{[2(\sigma_1^2+\sigma_2^2)]^{n/2}} \displaystyle\sum_{n=0}^{\infty} n(-e^{-\gamma})^{n-1}\left(\frac{1}{2\sigma_1^2}+n-1\right)^{-\frac{n}{2}}, & t>0 \\ \dfrac{1}{[2(\sigma_1^2+\sigma_2^2)]^{n/2}} \displaystyle\sum_{n=0}^{\infty} n(-e^{\gamma})^{n-1}\left(\frac{1}{2\sigma_2^2}+n+1\right)^{-\frac{n}{2}}, & t<0 \end{cases} \tag{2.40}$$

where the details may be found in Appendix B. Before proceeding to take the expected value of the terms in (A.21), (A.23) which are the final expressions for (2.13) and (2.14) a few definitions will be needed.

$$E\{\tilde{X}_k(0)\ \tilde{X}_k'(0)\} \triangleq P_{k|k}(0) \tag{2.41}$$

$$E\{\tilde{X}_k(1)\ \tilde{X}_k'(1)\} \triangleq P_{k|k}(1) \tag{2.42}$$

$$E\{\tilde{X}_k(0)\ \tilde{X}_k'(1)\} \triangleq S_k \tag{2.43}$$

Using the above definitions and the results obtained for the expected value of the likelihood ratio and its functions we proceed to take the

the expected value of (A.21) and (A.23) to result in the following equations for the gains $K_{k+1}^{(i)}$, i = 0,1,

$$0 = \frac{\partial}{\partial K_{k+1}^{(0)}} = -\tilde{\eta}_0 A_k P_{k|k}(0) A_k' C_{k+1}' - \tilde{\eta}_1 A_k S_k' A_k' C_{k+1}' - (\tilde{\eta}_0 + \tilde{\eta}_1)\theta \Gamma_k Q_k \Gamma_k' C_{k+1}'$$
$$+ \tilde{\eta}_0 K_{k+1}^{(0)}[C_{k+1} A_k P_{k|k}(0) A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}' + R_{k+1}]$$
$$+ \tilde{\eta}_1 K_{k+1}^{(1)}[C_{k+1} A_k S_k' A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}' + R_{k+1}]$$
$$+ \lambda\{-\tilde{\eta}_0 A_k P_{k|k}(0) A_k' C_{k+1}' - \tilde{\eta}_1 A_k S_k' A_k' C_{k+1}' - (\tilde{\eta}_0 + \tilde{\eta}_1)\theta \Gamma_k Q_k \Gamma_k' C_{k+1}'$$
$$+ \tilde{\eta}_0 K_{k+1}^{(0)}[C_{k+1} A_k P_{k|k}(0) A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}' + R_{k+1}]$$
$$+ \tilde{\eta}_1 K_{k+1}^{(1)}[C_{k+1} A_k S_k' A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}' + R_{k+1}]\} \qquad (2.44)$$

$$0 = \frac{\partial}{\partial K_{k+1}^{(1)}} = -\tilde{\eta}_1 A_k S_k A_k' C_{k+1}' - \tilde{\eta}_2 A_k P_{k|k}(1) A_k' C_{k+1}' - \tilde{\eta}_1 \theta \Gamma_k Q_k \Gamma_k' C_{k+1}'$$
$$- \tilde{\eta}_2 \theta \Gamma_k Q_k \Gamma_k' C_{k+1}' + \tilde{\eta}_0 K_{k+1}^{(0)}[C_{k+1} A_k S_k A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}'$$
$$+ R_{k+1}] + \tilde{\eta}_0 K_{k+1}^{(1)}[C_{k+1} A_k P_{k|k}(1) A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1} + R_{k+1}]$$
$$+ \lambda\{-\tilde{\eta}_1 A_k S_k A_k' C_{k+1}' - \tilde{\eta}_2 A_k P_{k|k}(1) A_k' C_{k+1}' - \tilde{\eta}_1 \theta \Gamma_k Q_k \Gamma_k' C_{k+1}'$$
$$- \tilde{\eta}_2 \theta \Gamma_k Q_k \Gamma_k' C_{k+1}'$$
$$+ \tilde{\eta}_0 K_{k+1}^{(0)}[C_{k+1} A_k S_k A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}' + R_{k+1}]$$
$$+ \tilde{\eta}_0 K_{k+1}^{(1)}[C_{k+1} A_k P_{k|k}(1) A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}' + R_{k+1}]\} \qquad (2.45)$$

After some algebra these expressions may be written compactly in the form.

$$\tilde{\eta}_0 K_{k+1}^{(0)} [a_1] + \tilde{\eta}_1 K_{k+1}^{(1)} [b_1] = [c_1] \tag{2.46}$$

$$\tilde{\eta}_0 K_{k+1}^{(0)} [a_2] + \tilde{\eta}_0 K_{k+1}^{(1)} [b_2] = [c_2] \tag{2.47}$$

where the following notation has been used.

$$a_1 \triangleq (C_{k+1}A_kP_{k|k}(0)A_k'C_{k+1}' + R_{k+1})(1+\lambda) + \theta_1 C_{k+1}\Gamma_k\theta_k\Gamma_k'C_{k+1}'$$
$$+ \lambda\,\theta_0\, C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}' \tag{2.48}$$

$$b_1 \triangleq (C_{k+1}A_kS_k'A_k'C_{k+1}' + R_{k+1})(1+\lambda) + \theta_1 C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}'$$
$$+ \theta_0 C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}' \tag{2.49}$$

$$c_1 \triangleq (-\tilde{\eta}_0 A_1 P_{k|k}(0)A_k'C_{k+1}' - \tilde{\eta}_1 A_kS_k'A_k'C_{k+1}')(1+\lambda) - (\tilde{\eta}_0 + \tilde{\eta}_1)\theta_1\Gamma_kQ_k\Gamma_k'C_{k+1}'$$
$$- \lambda(\tilde{\eta}_0 + \tilde{\eta}_1)\theta_0\Gamma_kQ_k\Gamma_k'C_{k+1}' \tag{2.50}$$

$$a_2 \triangleq (C_{k+1}A_kS_kA_k'C_{k+1}' + R_{k+1})(1+\lambda) + \theta_1 C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}' + \lambda\,\theta_0\, C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}' \tag{2.51}$$

$$b_2 \triangleq (C_{k+1}A_kP_k\ _k(1)A_k'C_{k+1}' + R_{k+1})(1+\lambda) + \theta_1 C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}' + \lambda\theta_0 C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}' \tag{2.52}$$

$$c_2 \triangleq (-\tilde{\eta}_1 A_kS_kA_k'C_{k+1}' - \tilde{\eta}_2 A_kP_{k|k}(1)A_k'C_{k+1}' - \tilde{\eta}_1\theta_1\Gamma_kQ_k\Gamma_k'C_{k+1}'$$
$$- \tilde{\eta}_2\theta_1\Gamma_k\Gamma_k'C_{k+1}')(1+\lambda) \tag{2.53}$$

Equations (2.46)-(2.47) are two equations with two unknown $K_{k+1}^{(i)}$, $i = 0,1$ which may be solved to yield

$$K_{k+1}^{(1)} = \{[c_2] - [c_1][a_1^{-1}][a_2]\}\{\tilde{\eta}_0[b_2] - \tilde{\eta}_1\,[b_1][a_1^{-1}][a_2]\}^{-1} \tag{2.54}$$

$$K_{k+1}^{(0)} = \{[c_1] - \tilde{\eta}_1\{K_{k+1}^{(1)}\}^{-1}\cdot\,[b_1]\}\,[\tilde{\eta}_0 a_1]^{-1} \tag{2.55}$$

Now $K_{k+1}^{(i)}$, i = 0,1 in (2.46)-(2.47) can be obtained explicitly by using the definitions (2.48)-(2.53) in (2.54)-(2.55). The expressions found above allow the recursive computation of the filter gains. However, they are given as a function of the convariances $P_{k|k}(i)$ and S_k, so that to completely determine the algorithm, the expressions for updating the covariances need to be derived. We start by deriving the one-step prediction covariances defined by

$$P_{k+1|k}(i) = E\{\tilde{X}_{k+1|k}(i)\ \tilde{X}'_{k+1|k}(i)\} \quad , \quad i=0,1 \tag{2.56}$$

where

$$\begin{aligned} \tilde{X}_{k+1|k}(i) &= X_{k+1} - A_k\hat{X}_{k|k}(i) \\ &= X_{k+1} - \hat{X}_{k+1|k}(i) \\ &= A_kX_k + \sqrt{\theta}\,\Gamma_k w_k - A_k\hat{X}_{k|k}(i) \end{aligned} \tag{2.57}$$

The substitution of (2.54) into (2.56) yields

$$\begin{aligned} P_{k+1|k}(i) &= E\{A_k\tilde{X}_{k|k}(i) + \sqrt{\theta_i}\,\Gamma_k w_k][A_k\tilde{X}_{k|k}(i) + \sqrt{\theta_i}\,\Gamma_k w_k]'\} \\ &= A_kE\{\tilde{X}_{k|k}(i)\tilde{X}'_{k|k}(i)\}A'_k + \theta_i\Gamma_kE\{w_kw'_k\}\Gamma'_k \\ &= A_kP_{k|k}(i)A'_k + \theta_i\Gamma_k\Gamma'_k, \qquad i=0,1 \end{aligned} \tag{2.58}$$

Where θ_0, θ_1 are the values of θ which maximizes the performance index in the two regions as specified in (2.6). We now form the following random variables $\tilde{X}_{k+1}(0)$ and $\tilde{X}_{k+1}(1)$, by using (2.7) as follows

$$\tilde{X}_{k+1}(i) = X_{k+1} - \hat{X}_{k+1}(i)$$

$$= A_k X_k + \sqrt{\theta_i}\,\Gamma_k w_k - A_k \hat{X}_k(i) - K_{k+1}^{(i)}[C_{k+1} A_k \tilde{X}_k(i) + \sqrt{\theta_i}\, C_{k+1} w_k + v_{k+1}],$$

$$i = 0,1 \qquad (2.59)$$

The substitution of $\tilde{X}_{k+1}(i)$, $i=0,1$ into the definition of S_{k+1} in (2.43) results after taking the expectations in the expressions.

$$S_{k+1} = A_k S_k A_k' + \theta \Gamma_k Q_k \Gamma_k' - (K_{k+1}^{(i)} + K_{k+1}^{(0)})(C_{k+1} A_k S_k A_k' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k')$$

$$+ K_{k+1}^{(0)}(C_{k+1} A_k S_k A_k' C_{k+1}' + \theta C_{k+1} \Gamma_k Q_k \Gamma_k' C_{k+1}' + R_{k+1}) K_{k+1}^{(1)'} \qquad (2.60)$$

The filter covariance error matrix P_{k+1} is defined by

$$P_{k+1} \triangleq E\{\tilde{X}_{k+1} \tilde{X}_{k+1}\} \qquad (2.61)$$

where

$$\tilde{X}_{k+1} \triangleq X_{k+1} - \hat{X}_{k+1}$$

$$= A_k X_k + \sqrt{\theta}\,\Gamma_k w_k - \frac{(\hat{X}_{k+1}(0) + \Lambda_{k+1} \hat{X}_{k+1}(1))}{1 + \Lambda_{k+1}} \qquad (2.62)$$

After some extensive algebra we find that (2.61) becomes

$$\tilde{X}_{k+1} = \{A_k \tilde{X}_k(0) + \Lambda_{k+1} A_k \tilde{X}_k(1) + (1+\Lambda_{k+1})\sqrt{\theta}\,\Gamma_k w_k - K_{k+1}^{(0)}(C_{k+1} A_k \tilde{X}_k(0)$$

$$+ \sqrt{\theta}\, C_{k+1} \Gamma_k w_k + v_{k+1}) - \Lambda_{k+1} K_{k+1}^{(1)}(C_{k+1} A_k \tilde{X}_k(1) + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k$$

$$+ v_{k+1})\}/(1 + \Lambda_{k+1}) \qquad (2.63)$$

On substitutive (2.63) into (2.61), taking the expectation, simplifying, and using the notations $\tilde{\eta}_0$, $\tilde{\eta}_1$, and $\tilde{\eta}_2$ one obtains

$$
\begin{aligned}
P_{k+1} = {} & \tilde{\eta}_0 P_{k+1|k}(0)[I - C'_{k+1}K^{(0)'}_{k+1}] + \tilde{\eta}_1 A_k S_k A'_k [I - C'_{k+1}K^{(1)}_{k+1}] \\
& + \tilde{\eta}_1 A_k S'_k A'_k (I - C'_{k+1}K^{(0)'}_{k+1}) + \tilde{\eta}_2 P_{k+1|k}(1)[I - C'_{k+1}K^{(1)'}_{k+1}] \\
& + (\tilde{\eta}_0 + \tilde{\eta}_1)^2 \theta \Gamma_k Q_k \Gamma'_k + (\tilde{\eta}_0 + \tilde{\eta}_1)\theta \Gamma_k Q_k \Gamma'_k C'_{k+1} K^{(0)'}_{k+1} - (\tilde{\eta}_1 + \tilde{\eta}_2)\theta \Gamma_k Q_k \Gamma'_k C'_{k+1} K^{(1)'}_{k+1} \\
& - \tilde{\eta}_0 K^{(0)}_{k+1} C_{k+1} P_{k+1|k}(0) - \tilde{\eta}_1 K^{(0)}_{k+1} C_{k+1} A_k S_k A'_k + \tilde{\eta}_0 K^{(0)}_{k+1}(C_{k+1} P_{k+1|k}(0) C'_{k+1}) K^{(0)'}_{k+1} \\
& + \tilde{\eta}_1 K^{(0)}_{k+1} C_{k+1} A_k S_k A'_k C'_{k+1} K^{(1)'}_{k+1} - (\tilde{\eta}_0 + \tilde{\eta}_1)\theta K^{(0)}_{k+1} \Gamma_k Q_k \Gamma'_k \\
& + \tilde{\eta}_0 \theta K^{(0)}_{k+1} C_{k+1} \Gamma_k Q_k \Gamma'_k K^{(0)'}_{k+1} - \tilde{\eta}_1 \theta K^{(0)}_{k+1} C_{k+1} \Gamma_k Q_k \Gamma'_k C'_{k+1} K^{(1)'}_{k+1} \\
& + \tilde{\eta}_0 K^{(1)}_{k+1} R_{k+1} K^{(0)'}_{k+1} + \tilde{\eta}_1 K^{(0)}_{k+1} R_{k+1} K^{(1)'}_{k+1} - \tilde{\eta}_1 K^{(1)}_{k+1} C_{k+1} A_k S'_k A'_k \\
& - \tilde{\eta}_2 K^{(1)}_{k+1} C_{k+1} P_{k+1|k}(1) + \tilde{\eta}_1 K^{(1)}_{k+1} C_{k+1} A_k S'_k A'_k C'_{k+1} K^{(0)}_{k+1} \\
& + \tilde{\eta}_2 K^{(1)}_{k+1} C_{k+1} P_{k+1|k}(1) C'_{k+1} K^{(1)'}_{k+1} \\
& - (\tilde{\eta}_1 + \tilde{\eta}_2)\theta K^{(1)}_{k+1} C_{k+1} \Gamma_k Q_k \Gamma'_k + \tilde{\eta}_1 \theta K^{(1)}_{k+1} C_{k+1} \Gamma_k Q_k \Gamma'_k C'_{K+1} K^{(0)'}_{k+1} \\
& + \tilde{\eta}_2 \theta K^{(1)}_{k+1} C_{k+1} \Gamma_k Q_k \Gamma'_k C'_{k+1} K^{(1)'}_{k+1} + \tilde{\eta}_1 K^{(1)}_{k+1} R_{k+1} K^{(0)'}_{k+1} + \tilde{\eta}_2 K^{(1)}_{k+1} R_{k+1} K^{(1)'}_{k+1} \qquad (2.64)
\end{aligned}
$$

Now, we have completed the derivation of all the expressions required for the computation of the recursive algorithm and evaluating its performance. The detailed computational procedure is discussed in the next chapter.

CHAPTER 3

COMPUTATION PROCEDURE

3.1. Introduction

In this chapter, the computation procedure of the recursive algorithm derived in Chapter 2, and other computational aspects will be discussed. In Section 3.2 the evolution of the recursive algorithm will be considered. Then, the initialization requirements will be given. Finally several simplifications in the computation are discussed.

3.2. Computation Procedure

Step 1. Initialization

In this part we must specify initial estimates to: $P_{k|k}(0)$, $P_{k\ k}(1)$, S_k, θ, θ_0, θ_1, and to the likelihood ratio Λ_k or more specifically its expected value.

Step 2. Computation of $P_{k+1|k}(0)$ and $P_{k+1|k}(1)$

The computation of the one step covariance error matrices $P_{k+1|k}(0)$ and $P_{k+1|k}(1)$ is based on their equations described by (2.61)-(2.62). The maximizing values of θ_0 and θ_1, initially will be the initial guesses specified in stage one. At all other times θ_0 and θ_1 will be the result of step 7 of the previous iteration.

Step 3: Computation of σ_1^2 and σ_2^2

The procedure of deriving σ_1^2 and σ_2^2 was given in detail in Appendix C. First, the mean β_0 is computed by using (2.26). It is a function of the following parameters

$$\beta_0 = f_0\{P_{k|k}(i),\ G_k(i),\ \theta\},\ i=0,1 \tag{3.1}$$

Then the matrices $G_k(i)$, i=0,1 are obtained from (2.20), which are a function of the following parameters

$$G_k(i) = f_1\{C_{k+1}, R_{k+1}, P_{k+1|k}(i)\}, \qquad i=0,1 \tag{3.2}$$

Initially $P_{k|k}(i)$, i=0,1 are known. In the computation of $P_{k+1|k}(i)$, i=0,1 θ is fixed to its true value. After β_0 and β_1 has been computed we can compute σ_1^2, σ_2^2 from (2.28) and (2.29).

Step 4: The evaluation of γ_k and the expected value of the likelihood-ratio

γ_k is described by Equation B.9 where initially we must use our estimate for $L_k \triangleq L_k(1) - L_k(0)$ which must have been specified in step 1, with all the other parameters known. The likelihood-ratio is computed via the recursive relation (B.75).

Step 5: Computation of $\tilde{\eta}_0$, $\tilde{\eta}_1$, $\tilde{\eta}_2$

Since the solution of $\tilde{\eta}_0$, $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are given in terms of an infinite series. It is more convenient to use numerical integration, a 32 points Gaussian quadrature routine for example. It is especially appealing for the case of n = 2 since then the limits of $[0,\infty]$ can be changed to [0,1] via a change of variables, otherwise the original integral

$$\int_0^\infty \frac{x^{\gamma-1}e^{-\mu x}}{(1-\beta e^{-x})^2} \tag{3.3}$$

can be computed by placing an artificial upper limit on (3.3) by recognizing that the term that controls the convergence of (3.3) is $e^{-\mu x}$. Therefore let:

$$e^{-\mu x} = 10^{-\alpha} \tag{3.4}$$

where α is an arbitrarily chosen desired number sufficiently large in order to ensure convergence. Once α is chosen x can be computed to be the upper limit of (3.3).

<u>Step 6</u>: Computation of $K_{k+1}^{(i)}$, i=0,1

$K_{k+1}^{(i)}$, i=0,1 are computed using equations (2.54)-(2.55). It should be noted that in this stage θ is fixed to its true value.

<u>Step 7</u>: Determination of $\max\{P_{k+1}$ i=0,1 for $\theta \in H$

Using the results of step 6 we compute the value of $P_{k+1}(i)$, i=0,1 corresponding to max $\theta \in H_i$. The procedure is illustrated for $P_{k+1}(0)$. We first substitute the following

$$P_{k+1}\Big|_{\theta=\theta_0},\; P_{k+1|k}^{(1)}\Big|_{\theta=\theta_0}$$ into (2.64) and then find $\theta=\theta_0$

for which (2.64) is maximized. Fortunately, since $P_{k+1}(i)$ is linearly related to θ we can set max θ_i to be at its boundry values. As a result of this step we have the value of $P_{k+1}(i)$ for the next iteration, and the values θ_0, θ_1 to be used in step 2 of the next iteration.

<u>Step 8</u>: The update of S_{k+1}

The update of S_{k+1} is performed via Equation (2.60). Note that in this step θ is fixed to its true value. At that point one iteration has been completely finished, and we start again in Step 2.

3.3. On-Line Computation Procedure

In section 3.2 the necessary steps for analyzing the performance of the algorithm were described. However, during the computation procedure when measurements are available the following changes are necessary.

In step 4 of the previous section it is no longer necessary to compute the expected values of the likelihood-ratio and its functions η_0, η_1 and η_2. Instead we evaluate Λ_{k+1} using the sequential likelihood ratio formula [16] given by (2.20)-(2.21). The likelihood formula together with the derived gains (2.54), (2.55) are then used in the expressions of the filter (2.7) to obtain the desired estimates. It should be noted that in the above mode the observations and the equations for $\hat{X}_{k+1}(i)$, i=0,1 and $\hat{X}_{k+1}$ should be included appropriately.

CHAPTER 4

SUMMARY AND CONCLUSIONS

In this thesis a joint detection-estimation scheme has been used to derive an on line recursive algorithm for a class of systems with uncertain noise parameters. The algorithm derived is most suitable for a wide class of systems for which the uncertainty about the unknown parameter appears in the form of bounds. It is also capable of handling cases where the unknown parameters are time-varying as well as constant. The derived scheme is only stage by stage optimal and not globally optimal, therefore, relieving some of the computational requirements such as memory requirements and computational speed. The algorithm is capable of approximately computing the performance of the system rather than simulation.

Since observations are available in the on-line filtering mode we can compute the likelihood-ratio and its functions via the sequential likelihood-ratio formula instead of its expected value. The gains of the system are computed directly by the algorithm and may in general depend on the observations. It should be noted that in the above mode the observations and the equations for $\hat{X}_{k+1}(i)$, $i=0,1$ should be included appropriately. Both off-line computations of performance and simulations have been made to demonstrate the information flow in the filter.

APPENDIX A

THE DERIVATION OF $K_{k+1}^{(0)}$ and $K_{k+1}^{(1)}$

The purpose of this appendix is to outline a detailed derivation of $K_{k+1}^{(1)}$ and $K_{k+1}^{(1)}$ up to the point where we need to take the expected value. The scalar cost functional $\tilde{C}_{k+1}$ to be minimized was given in (2.12) which may be written as

$$\tilde{C}_{k+1} = \max_{\theta \in H_1} C_{k+1}(\theta) + \lambda \max_{\theta \in H_0} C_{k+1}(\theta) \tag{A.1}$$

For convenience and because of the symmetry of the problem only the first term in (A.1) will be explicitly derived. From (A.1) and (2.12) the expression for $C_{k+1}(\theta)$ may be explicitly written as:

$$C_{k+1}(\theta) = E\left\{X_{k+1}'X_{k+1} - 2X_{k+1}' \frac{(\hat{X}_{k+1}(0) + \Lambda_{k+1}\hat{X}_{k+1}(1))}{1+\Lambda_{k+1}}\right.$$

$$\left. + \frac{(\hat{X}_{k+1}(0) + \Lambda_{k+1}\hat{X}_{k+1}(1))'}{(1+\Lambda_{k+1})} \frac{(\hat{X}_{k+1}(0) + \Lambda_{k+1}\hat{X}_{k+1}(1))}{(1+\Lambda_{k+1})}\right\} \tag{A.2}$$

The substitution of (2.7) into (A.2) results in

$$C_{k+1}(\theta) = E\left\{X_{k+1}'X_{k+1} - 2X_{k+1}'\left\{A_k\hat{X}_k(0) + K_{k+1}^{(0)}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))\right.\right.$$

$$\left. + \Lambda_{k+1}(A_k\hat{X}_k(1) + K_{k+1}^{(1)}(y_{k+1} - C_{k+1}A_k\hat{X}_k(1)))\right\}/(1+\Lambda_{k+1})$$

$$+ \left\{A_k\hat{X}_k(0) + K_{k+1}^{(0)}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0)) + \Lambda_{k+1}(A_k\hat{X}_k(1)\right.$$

$$\left.\left. + K_{k+1}^{(1)}(y_{k+1} - C_{k+1}A_k\hat{X}_k(1)))\right\}/(1+\Lambda_{k+1})^2\right\} \tag{A.3}$$

We now need to take the derivatives of (A.3) (after maximizing with respect to θ) with respect to $K_{k+1}^{(0)}$ and $K_{k+1}^{(1)}$. Again only one expression will be derived due to the symmetry of the problem. We shall use the following results from linear algebra

$$\frac{\partial}{\partial K}(x'Ky) = xy' \tag{A.4}$$

$$\frac{\partial}{\partial K}(y'Kx) = yx' \tag{A.5}$$

$$\frac{\partial}{\partial K}(x'K'Ky) = K(xy' + yx') \tag{A.6}$$

where K is an nxn matrix and x,y are n and m column vectors. The use of these relations in taking the derivatives of (A.3) with respect to $K_{k+1}^{(0)}$ and $K_{k+1}^{(1)}$ results in

$$\begin{aligned}\frac{\partial C_{k+1}(0)}{\partial K_{k+1}^{(0)}} = E\Big\{ &- \frac{2X_{k+1}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))'}{1+\Lambda_{k+1}} + \frac{2A_k\hat{X}_k(0)(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))'}{(1+\Lambda_{k+1})^2} \\ &+ \frac{2K_{k+1}^{(0)}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))'}{(1+\Lambda_{k+1})^2} \\ &+ \frac{2\Lambda_{k+1}\hat{X}_{k+1}(1)(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))'}{(1+\Lambda_{k+1})^2}\Big\}\end{aligned} \tag{A.7}$$

After some algebra it may be further reduced to

$$\begin{aligned}\frac{\partial C_{k+1}(0)}{\partial K_{k+1}^{(0)}} = E\{\{&- 2X_{k+1}(1+\Lambda_{k+1}) + 2A_k\hat{X}_k(0) + 2K_{k+1}^{(0)}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0)) \\ &+ 2\Lambda_{k+1}\hat{X}_{k+1}(1)\}\{(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))'\}\}/(1+\Lambda_{k+1})^2\end{aligned} \tag{A.8}$$

The substitution of (1.4) and (2.7) for i=1 into (A.8) results in:

$$\frac{\partial C_{k+1}(0)}{\partial K_{k+1}^{(0)}} = E\{[-2(1+\Lambda_{k+1})A_k X_k - 2(1+\Lambda_{k+1})\sqrt{\theta}\,\Gamma_k w_k + 2A_k\hat{X}_k(0)$$

$$+ 2K_{k+1}^{(0)}(y_{k+1}-C_{k+1}A_k\hat{X}_k(0)) + 2\Lambda_{k+1}A_k\hat{X}_k(1) + 2\Lambda_{k+1}K_{k+1}^{(1)}\cdot$$

$$(y_{k+1}-C_{k+1}A_k\hat{X}_k(1))]\cdot[(y_{k+1}-C_{k+1}A_k\hat{X}_k(0))']\}/(1+\Lambda_{k+1})^2 \tag{A.9}$$

In order to perform the expectation, the expression (A.9) is further simplified in terms of the error estimates defined as

$$\tilde{X}_k = X_k - \hat{X}_k \tag{A.10}$$

and the system noise w_k and v_{k+1}. The resulting final form of (A.9) becomes

$$\frac{\partial C_{k+1}(\theta)}{\partial K_{k+1}^{(0)}} = -\eta_0 A_k\tilde{X}_k(0)\tilde{X}_k'(0)A_k'C_{k+1}' - \eta_1 A_k\tilde{X}_k(1)\tilde{X}_k'(0)A_k'C_{k+1}'$$

$$-(\eta_0+\eta_1)\sqrt{\theta}\,\Gamma_k w_k\tilde{X}_k'(0)A_k'C_{k+1}' - \eta_0\sqrt{\theta}\,A_k X_k(0)w_k'\Gamma_k'C_{k+1}'$$

$$-\eta_1\sqrt{\theta}\,A_k\tilde{X}_k(1)w_k'\Gamma_k'C_{k+1}' - (\eta_0+\eta_1)\theta\Gamma_k w_k w_k'\Gamma_k'C_{k+1}'$$

$$-\eta_0 A_k\tilde{X}_k(0)v_{k+1}' - \eta_1 A_k\tilde{X}_k(1)v_{k+1}' - (\eta_0+\eta_1)\sqrt{\theta}\,\Gamma_k w_k v_{k+1}'$$

$$+\eta_0 K_{k+1}^{(0)}[C_{k+1}A_k\tilde{X}_k(0)\tilde{X}_k'(0)A_k'C_{k+1}' + \sqrt{\theta}\,C_{k+1}\Gamma_k w_k\tilde{X}_k'(0)A_k'C_{k+1}'$$

$$+ v_{k+1}\tilde{X}_k'(0)A_k'C_{k+1}' + \sqrt{\theta}\,C_{k+1}A_k\tilde{X}_k(0)w_k'\Gamma_k'C_{k+1}' + \theta C_{k+1}\Gamma_k w_k w_k'\Gamma_k'C_{k+1}'$$

$$+\sqrt{\theta}v_{k+1}w_k'\Gamma_k'C_{k+1}' + C_{k+1}A_k\tilde{X}_k(0)v_{k+1}' + \sqrt{\theta}C_{k+1}\Gamma_k w_k v_{k+1}' + v_{k+1}v_{k+1}']$$

$$+\eta_2 K_{k+1}^{(1)}[C_{k+1}A_k\tilde{X}_k(1)\tilde{X}_k'(0)A_k'C_{k+1}' + \sqrt{\theta}C_{k+1}\Gamma_k w_k\tilde{X}_k'(0)A_k'C_{k+1}'$$

$$+ v_{k+1}\tilde{X}_k'(0)A_k'C'_{k+1} + \sqrt{\theta}\, C_{k+1}A_k\tilde{X}_k(1)w_k'\Gamma_k'C_{k+1}' + \theta C_{k+1}\Gamma_k w_k w_k'\Gamma_k'C_{k+1}'$$

$$+ \sqrt{\theta}\, v_{k+1}w_k'\Gamma_k'C_{k+1}' + C_{k+1}A_k\tilde{X}_k(1)v_{k+1}' + \sqrt{\theta}C_{k+1}\Gamma_k w_k v_{k+1}' + v_{k+1}v_{k+1}']$$

$$\text{(A.11)}$$

where we have denoted η_0, η_1, and η_2 as defined in (2.15)-(2.17). In a similar fashion the expression for $\frac{\partial C_{k+1}(\theta)}{\partial K_{k+1}^{(1)}}$ can be obtained to result in

$$\frac{\partial C_{k+1}(\theta)}{\partial K_{k+1}^{(1)}} = -\eta_1 A_k\tilde{X}_k(0)\tilde{X}_k'(1)A_k'C_{k+1}' - \eta_2 A_k\tilde{X}_k(1)\tilde{X}_k'(1)A_k'C_{k+1}'$$

$$- \sqrt{\theta}\eta_1\, \Gamma_k w_k\tilde{X}_k'(1)A_k'C_{k+1}' - \eta_2\sqrt{\theta}\, \Gamma_k w_k\tilde{X}_k'(1)A_k'C_{k+1}'$$

$$- \eta_1\sqrt{\theta}\, A_k\tilde{X}_k(1)w_k'\Gamma_k'C_{k+1}' - \eta_2\sqrt{\theta}\, A_k\tilde{X}_k(1)w_k'\Gamma_k'C_{k+1}'$$

$$- \eta_1\theta\Gamma_k w_k w_k'\Gamma_k'C_{k+1}' - \eta_2\theta\Gamma_k w_k w_k'\Gamma_k'C_{k+1}'$$

$$- \eta_1 A_k\tilde{X}_k(0)v_{k+1}' - \eta_2 A_k\tilde{X}_k(1)v_{k+1}' - \sqrt{\theta}\,\eta_1\Gamma_k w_k v_{k+1}' - \eta_2\sqrt{\theta}\,\Gamma_k w_k v_{k+1}'$$

$$+ \eta_1 K_{k+1}^{(0)}(C_{k+1}A_k\tilde{X}_k(0)\tilde{X}_k'(1)A_k'C_{k+1}' + \sqrt{\theta}\, C_{k+1}\Gamma_k w_k\tilde{X}_k'(1)A_k'C_{k+1}'$$

$$+ v_{k+1}'\tilde{X}_{k+1}'A_k'C_{k+1}' + \sqrt{\theta}\, C_{k+1}A_k\tilde{X}_k(0)\, w_k'\Gamma_k'C_{k+1}' + \theta C_{k+1}\Gamma_k w_k w_k'\Gamma_k'C_{k+1}'$$

$$+ \sqrt{\theta}\, v_{k+1}\, w_k'\Gamma_k'C_{k+1}' + C_{k+1}A_k\tilde{X}_k(0)v_{k+1}' + \sqrt{\theta}\, C_{k+1}\Gamma_k w_k v_{k+1}' + v_{k+1}v_{k+1}')$$

$$+ \eta_0 K_{k+1}^{(1)}[C_{k+1}A_k\tilde{X}_k(1)\tilde{X}_k'(1)A_k'C_{k+1}' + \sqrt{\theta}\, C_{k+1}\Gamma_k w_k\tilde{X}_k'(1)A_k'C_{k+1}'$$

$$+ v_{k+1}\tilde{X}_k(1)A_k'C_{k+1}'$$

$$+ \sqrt{\theta}\, C_{k+1} A_k \tilde{X}_k(1) w_k' \Gamma_k' C_{k+1}' + \theta\, C_{k+1} \Gamma_k w_k w_k' \Gamma_k' C_{k+1}' + \sqrt{\theta}\, v_{k+1} w' \Gamma_k' C_{k+1}'$$

$$+ C_{k+1} A_k \tilde{X}_k(1) v_{k+1}' + \sqrt{\theta}\, C_{k+1} \Gamma_k w_k v_{k+1}' + v_{k+1} v_{k+1}'] \qquad (A.12)$$

These are the expressions to be used in (2.13) and (2.14).

APPENDIX B

THE EVALUATION OF THE EXPECTED VALUE OF THE LIKELIHOOD-RATIO

We have shown before that detection and estimation are highly related to each other via the sequential likelihood ratio formula. Using the formula given in [16] pp. 157-161 let the natural log of the likelihood ratio be:

$$L_{k+1} \triangleq \ell n(\Lambda_{k+1}) \qquad (B.1)$$

$\therefore$

The sequential likelihood ratio is given by the following formula

$$L_{k+1} = \ell n\left\{\frac{\det(C_{k+1}P_{k+1|k}(0)C'_{k+1} + R_{k+1})}{\det(C_{k+1}P_{k+1|k}(1)C'_{k+1} + R_{k+1})}\right\} + L_k$$
$$+ \frac{1}{2}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))'(C_{k+1}P_{k+1|k}(0)C'_{k+1} + R_{k+1})^{-1}(y_{k+1} - C_{k+1}A_k\hat{X}_k(0))$$
$$- \frac{1}{2}(y_{k+1} - C_{k+1}A_k\hat{X}_k(1))'(C_{k+1}P_{k+1|k}(1)C'_{k+1} + R_{k+1})^{-1}(y_{k+1} - C_{k+1}A_k\hat{X}_k(1)) \qquad (B.2)$$

Now (B.2) may be expressed in terms of the estimation errors $\tilde{X}_k(i) = X_k - \hat{X}_k(i)$, $i = 0,1$ by substituting into (B.2) the expressions (1.2) and (1.3) for $\hat{X}_{k+1}$ and y_{k+1}. After some algebraic manipulation we obtain

$$L_{k+1} = \ell n\left\{\frac{\det(C_{k+1}P_{k+1|k}(0)C'_{k+1} + R_{k+1})}{\det(C_{k+1}P_{k+1|k}(1)C'_{k+1} + R_{k+1})}\right\} + L_k$$
$$+ \frac{1}{2}(C_{k+1}A_k\tilde{X}_k(0)+ \sqrt{\theta}\; C_{k+1}\Gamma_k\omega_k+v_{k+1})'(C_{k+1}P_{k+1|k}(0)C'_{k+1}+R_{k+1})^{-1}(C_{k+1}A_k\tilde{X}_k(0)$$
$$+ \sqrt{\theta}\; C_{k+1}\Gamma_k\omega_k+v_{k+1})$$

$$- \frac{1}{2}(C_{k+1}A_k\tilde{X}_k(1) + \sqrt{\theta}\ C_{k+1}\Gamma_k\omega_k + v_{k+1})'(C_{k+1}P_{k+1|k}(1)C'_{k+1} + R_{k+1})^{-1}$$

$$\cdot(C_{k+1}A_k\tilde{X}_k(1) + \sqrt{\theta}\ C_{k+1}\Gamma_k\omega_k + v_{k+1}) \tag{B.3}$$

It should be noted that the first two terms are constants, as far as the expectation is concerned, and may be denoted by:

$$\gamma_k \triangleq \ell n\left\{\frac{\det(C_{k+1}P_{k+1|k}(0)C'_{k+1} + R_{k+1})}{\det(C_{k+1}P_{k+1|k}(1)C'_{k+1} + R_{k+1})}\right\} + L_k \tag{B.4}$$

Furthermore, the following notations are used

$$G_k(i) = (C_{k+1}P_{k+1|k}(i)C'_{k+1} + R_{k+1})^{-1} \quad , \; i = 0,1 \tag{B.5}$$

$$X_k(i) = (C_{k+1}A_k\tilde{X}_k(i) + \sqrt{\theta}\ C_{k+1}\Gamma_k\omega_k + v_{k+1})'G_k(i)(C_{k+1}A_k\tilde{X}_k(i) + \sqrt{\theta}\ C_{k+1}\Gamma_k\omega_k + v_{k+1})$$

$$i = 0,1 \tag{B.5a}$$

The resulting expression for the likelihood ratio becomes then

$$L_{k+1} = \gamma_k + \frac{1}{2}X'_k(0)G_k(0)X_k(0) - \frac{1}{2}X'_k(1)G_k(1)X_k(1)$$

$$\equiv \gamma_k + Z \tag{B.6}$$

where

$$Z = \frac{1}{2}(X'_k(0)G_k(0)X_k(0) - X'_k(1)G_k(1)X_k(1))$$

so that the likelihood ratio is

$$\Lambda_{k+1} = \exp\{\gamma_k + Z\} \tag{B.7}$$

Since ω_k, v_{k+1} and X_k are Gaussian r.v then the random variable Z is a quadratic expression in variables which are conditionally Gaussian. Consequently, the conditional density of Z_k (over which the expectation is

to be performed) is a generalized chi-square density function [17]. However, it is not possible to obtain a closed form expression to the density, hence an approximate model is used. Since Z is a difference of two positive semidefinite quadratic forms of n Gaussian variables, we approximate its density by a convolution of two density functions each of which is chi-square with n degrees of freedom, one with positive r.v and one with negative r.v. This means that the density of Z is given by

$$f_Z(t) = f_1(z) * f_2(z) \tag{B.8}$$

where

$$f_1(z) = \begin{cases} \dfrac{1}{2^{n/2}\sigma_1^n \Gamma(\frac{n}{2})} \exp(-\dfrac{z}{2\sigma_1^2})(z)^{\frac{n}{2}-1}, & z > 0 \\ 0, & z \leq 0 \end{cases} \tag{B.9}$$

$$f_2(z) = \begin{cases} \dfrac{1}{2^{n/2}\sigma_2^n \Gamma(\frac{n}{2})} \exp(\dfrac{z}{2\sigma_2^2})(-z)^{\frac{n}{2}-1}, & z < 0 \\ 0, & z \geq 0 \end{cases} \tag{B.10}$$

In this approximation σ_1^2 and σ_2^2 are parameters to be determined from the moments of the random variable Z. The explicit expression for the convolution may be written as:

$$f_Z(t) = \frac{1}{2^n(\sigma_1\sigma_2)^n\Gamma^2(\frac{n}{2})} \int_{\max(0,t)}^{\infty} e^{-z/2\sigma_1^2} (z)^{\frac{n}{2}-1} e^{-\frac{(z-t)}{2\sigma_2^2}} (z-t)^{\frac{n}{2}-1} U(z-t)dz$$

$$= \frac{e^{t/2\sigma_2^2}}{(2\sigma_1\sigma_2)^n\Gamma^2(\frac{n}{2})} \int_{\max(0,t)}^{\infty} \exp\{-z(\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2})\}(z)^{\frac{n}{2}-1}(z-t)^{\frac{n}{2}-1} dz \tag{B.11}$$

two cases need to be considered.

For $t < 0$ we have

$$f_Z(t) = \frac{e^{t/2\sigma_2^2}}{(2\sigma_1\sigma_2)^n\Gamma^2(\frac{n}{2})}\int_0^\infty \exp\{-z(\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2})\}(z)^{\frac{n}{2}-1}(z-t)^{\frac{n}{2}-1}dz. \tag{B.12}$$

However, from [19] the following integral is used.

$$\int_0^\infty x^{\nu-1}(x+\beta)^{-Q}e^{-\mu x}dx = \frac{1}{\sqrt{\pi}}(\frac{\beta}{\mu})^{\nu-\frac{1}{2}}\exp\{\frac{\beta\mu}{2}\}\Gamma(\nu)K_{\frac{1}{2}-\nu}(\frac{\beta\mu}{2}) \tag{B.13}$$

where K_ν is the modified Bessel function and

$$|\arg\beta| < \pi,\ \mathrm{Re}\mu > 0,\ \mathrm{Re}\nu > 0$$

For our case in (B.13) we have

$$\mu = \frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2} \tag{B.14a}$$

$$Q = 1 - \frac{n}{2} \tag{B.14b}$$

$$\nu = \frac{n}{2} \tag{B.14c}$$

$$\beta = -t \tag{B.14d}$$

so that the solution to (B.13) becomes

$$f_Z(t) = \frac{\exp\{t/2\sigma_2^2\}}{(2\sigma_1\sigma_2)^n \Gamma^2(\frac{n}{2})} \frac{1}{\sqrt{\pi}} \left[-t\left(\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)\right]^{\frac{n}{2} - \frac{1}{2}} \exp\left\{-t\left(\frac{\sigma_1^2 - \sigma_2^2}{4\sigma_1^2\sigma_2^2}\right)\right\}$$

$$\cdot\, \Gamma(\tfrac{n}{2}) K_{\frac{1}{2} - \frac{n}{2}} \left\{-t\left(\frac{\sigma_1^2 + \sigma_2^2}{4\sigma_1^2\sigma_2^2}\right)\right\} \qquad , \ t < 0 \ . \qquad \text{(B.15)}$$

Equation (B.15) may be simplified by using the asymptotic expansion of the Bessel function [20].

$$K_\nu(Z) = \left(\frac{\pi}{2Z}\right)^{\frac{1}{2}} e^{-Z}\left\{1 + \frac{\mu-1}{8Z} + \frac{(\mu-1)(\mu-9)}{2!\ \ (8Z)^2} \ldots\right\} \qquad \text{(B.16)}$$

where ν fixed, $|Z|$ large and $\mu = 4\nu^2$. After substituting $\nu = \frac{n}{2} - \frac{1}{2}$, and $|Z| = -\frac{\beta\mu}{2} = t\left(\frac{\sigma_1^2 + \sigma_2^2}{4\sigma_1^2\sigma_2^2}\right)$ into (B.16) we take only the first term of the resulting asymptotic expansion of the Bessel function and substitute it into (B.15) to obtain the following:

$$f_Z(t) \simeq \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}} (-t)^{\frac{n}{2} - 1} \exp\left\{\frac{t}{2\sigma_2^2}\right\} \quad , \quad t < 0 \qquad \text{(B.17a)}$$

Similarly for $t > 0$ we have

$$f_Z(t) = \frac{\exp\left(\frac{t}{2\sigma_2^2}\right)}{2^n(\sigma_1\sigma_2)^n \Gamma^2(\frac{n}{2})} \int_t^\infty \exp\left\{-z\frac{(\sigma_1^2 + \sigma_2^2)}{2\sigma_1^2\sigma_2^2}\right\}(z)^{\frac{n}{2} - 1} (z-t)^{\frac{n}{2} - 1} dz \qquad \text{(B.17b)}$$

From [19] and [21] the general solution to (B.12) is obtained from the following integral.

$$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1}\exp(-\beta x)dx = \frac{1}{\sqrt{\pi}}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}}\Gamma(\mu)\exp\left(\frac{-\beta u}{2}\right)K_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right) \tag{B.18}$$

with

$\{\text{Re } \mu > 0, \text{ Re } \beta u > 0\}$ to result in

$$f_Z(t) = \frac{\exp\left(\frac{t}{2\sigma_2^2}\right)}{2^n(\sigma_1\sigma_2)^n\Gamma^2\left(\frac{n}{2}\right)}\frac{1}{\sqrt{\pi}}\left[t\cdot\frac{(2\sigma_1^2\sigma_2^2)}{\sigma_1^2+\sigma_2^2}\right]^{\frac{n}{2}-\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)\exp\left\{-t\frac{(\sigma_1^2+\sigma_2^2)}{4\sigma_1^2\sigma_2^2}\right\}$$

$$\cdot K_{\frac{n}{2}-\frac{1}{2}}\left\{t\frac{(\sigma_1^2+\sigma_2^2)}{4\sigma_1^2\sigma_2^2}\right\} \qquad , \; t > 0 \tag{B.19}$$

Following the same approach used for the case of $t < 0$, the substitution of the asymptotic expression of the Bessel function into (B.19) results in

$$f_Z(t) \cong \frac{1}{[2(\sigma_1^2+\sigma_2^2)]^{\frac{n}{2}}}(t)^{\frac{n}{2}-1}\exp\left\{\frac{-t}{2\sigma_1^2}\right\} \quad , \; t > 0 \tag{B.20}$$

Thus $f_Z(t)$ is obtained approximately by (B.12) and (B.20). We can now proceed to find the required expected values of the functions of the likelihood-ratio η_0, η_1 and η_2 defined by (2.15)-(2.17). The expected value of η_0 can be evaluated as follows:

$$E\{\eta_0\} = \int_{-\infty}^{\infty} \frac{1}{(1 + e^{\gamma}e^{t})^2} f_Z(t)dt = \int_{-\infty}^{0} (\cdot)dt + \int_{0}^{\infty} (\cdot)dt$$

$$\equiv \tilde{\eta}_0^- + \tilde{\eta}_0^+ \tag{B.21}$$

The first term $\tilde{\eta}_0^-$ may be written as

$$\tilde{\eta}_0^- = \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_{-\infty}^{0} \frac{(-t)^{\frac{n}{2} - 1}}{(1 + e^{\gamma}e^{t})^2} \exp\{\frac{t}{2\sigma_2^2}\} dt$$

$$= \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_{0}^{\infty} \frac{(t)^{\frac{n}{2} - 1}}{(1 + e^{\gamma}e^{-t})^2} \exp\{\frac{-t}{2\sigma_2^2}\} dt$$

$$\equiv C_1 \int_{0}^{\infty} \frac{x^{\nu-1}e^{-\mu x}}{(1 - \beta e^{-x})^2} dx \tag{B.22}$$

where C_1 is analogous to the constant term in front of the integral. In order to evaluate (B.22) let $|\beta| < 1$ and use a geometric series expansion to reduce it to the following form:

$$I_1 = \int_{0}^{\infty} \frac{x^{\nu-1}e^{-\mu x}}{(1 - \beta e^{-x})^2} dx = \int_{0}^{\infty} x^{\nu-1}e^{-\mu x} \sum_{n=0}^{\infty} \mu(\beta e^{-x})^{n-1} dx \tag{B.23}$$

which after interchanging summation and integration becomes

$$I_1 = \sum_{n=0}^{\infty} n\beta^{n-1} \int_0^{\infty} x^{\nu-1} e^{-x(\mu + n-1)} dx$$

$$= \sum_{n=0}^{\infty} n\beta^{n-1} (\mu + n-1)^{-\nu} \Gamma(\nu) \tag{B.24}$$

This expression is used in (B.22) by recognizing that: $\beta = -e^{\gamma}$, $\mu = \frac{1}{2\sigma_2^2}$ and $\nu = \frac{n}{2}$ to obtain the final result for $\tilde{\eta}_0^-$

$$\tilde{\eta}_0^- = \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}} \sum_{n=0}^{\infty} n(-e^{\gamma})^{n-1}\left(\frac{1}{2\sigma_2^2} + n-1\right)^{-\frac{n}{2}} \tag{B.25}$$

Similarly for the second term of (B.21)

$$\tilde{\eta}_0^+ = \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_0^{\infty} \frac{(t)^{\frac{n}{2}-1}}{(1 + e^{\gamma}e^{t})^2} \exp\{\frac{-t}{2\sigma_1^2}\} dt \tag{B.26}$$

which may be reduced to

$$\tilde{\eta}_0^+ = \frac{e^{-2\gamma}}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}\Gamma(\frac{n}{2})} \int_0^{\infty} \frac{(t)^{\frac{n}{2}-1}}{(1 + e^{-\gamma}e^{-t})^2} \exp\{-t(2 + \frac{1}{2\sigma_1^2})\} dt \tag{B.27}$$

Again recognizing that $\beta = e^{-\gamma}$, $\mu = 2 + \frac{1}{2\sigma_1^2}$ and $\nu = \frac{n}{2}$ and applying (B.22) the final answer is obtained

$$\tilde{\eta}_0^+ = \frac{e^{-2\gamma}}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}} \sum_{n=0}^{\infty} n(-e^{-\gamma})^{n-1}(1 + n + \frac{1}{2\sigma_1^2})^{-\frac{n}{2}} \tag{B.28}$$

Equations (B.25) and (B.28) are combined to yield the final expression for the expectation

$$\tilde{\eta}_0 = \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}\Gamma(\frac{n}{2})} \{ \sum_{n=0}^{\infty} n(-e^{-\gamma})^{n-1}(1 + n + \frac{\ell}{2\sigma_1^2})^{-\frac{n}{2}}$$

$$+ \sum_{n=0}^{\infty} n(-e^{\gamma})^{n-1}(n - 1 + \frac{1}{2\sigma_2^2})^{-\frac{n}{2}}\} \tag{B.29}$$

Following the same procedure outlined in (B.22)-(B.29) we find that

$$\tilde{\eta}_1 = \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}\Gamma(\frac{n}{2})} \{e^{-\gamma} \int_0^{\infty} \frac{(t)^{\frac{n}{2}-2}}{(1 + e^{-\gamma}e^{-t})^2} \exp\{-t(\frac{1}{2\sigma_1^2} + 1)\} \, dt$$

$$+ e^{\gamma} \int_0^{\infty} \frac{(t)^{\frac{n}{2}-1}}{(1 + e^{\gamma}e^{-t})^2} \exp\{-t(1 + \frac{1}{2\sigma_2^2})\}dt\}$$

$$= \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}} \{ \sum_{n=0}^{\infty} n(-e^{-\gamma})^{n-1}(\frac{1}{2\sigma_1^2} + n-1)^{-\frac{n}{2}}$$

$$+ \sum_{n=0}^{\infty} n(-e^{\gamma})^{n-1}(\frac{1}{2\sigma_2^2} + n+1)^{-\frac{n}{2}} \} \tag{B.30}$$

$$\tilde{\eta}_2 = \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}\Gamma(\frac{n}{2})} \{ \int_0^\infty \frac{(t)^{\frac{n}{2}-1}}{(1 + e^{-\gamma}e^{-t})^2} \exp \{\frac{-t}{2\sigma_1^2}\}dt$$

$$+ e^{2\gamma} \int_0^\infty \frac{(t)^{\frac{n}{2}-1}}{(1 + e^{\gamma}e^{-t})^2} \exp \{t(\frac{1}{2\sigma_2^2} + 2)\}dt\}$$

$$= \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}}\Gamma(\frac{n}{2})} \{ \sum_{n=0}^{\infty} n(-e^{-\gamma})^{n-1}(\frac{1}{2\sigma_1^2} + n-1)^{-\frac{n}{2}}$$

$$+ \sum_{n=0}^{\infty} n(-e^{-\gamma})^{n-1}(\frac{1}{2\sigma_2^2} + n+1)^{-\frac{n}{2}} \tag{B.31}$$

where the same procedure of (B.22)-(B.29) has been used.

The expected value of Λ_{k+1} is needed mainly for step 4 of the computation of the algorithm (see Chapter 3, Section 2). We have denoted

$$\Lambda_{k+1} = e^{\gamma_k + Z} \tag{B.32}$$

where the term e^{γ_k} is constant as far as the expectation is concerned and e^Z is observation dependent. Therefore, (B.32) can be described by the following recursion

$$\Lambda_{k+1} = \Lambda_k C_k e^Z \tag{B.33}$$

where Λ_k and C_k are the first and second terms in (2.20) respectively. Using the results given by [19] we obtain

$$\tilde{\eta} = E\{\Lambda_{k+1}\} = \Lambda_k C_k \frac{1}{[2(\sigma_1^2 + \sigma_2^2)]^{\frac{n}{2}} \Gamma(\frac{n}{2})} \{\frac{1}{(1 + \frac{1}{2\sigma_2^2})^{\frac{n}{2}}} + \frac{1}{(\frac{1}{2\sigma_1^2} - 1)^{\frac{n}{2}}}\}$$

APPENDIX C

THE DERIVATION OF THE MEAN AND VARIANCE OF $f_Z(t)$

The purpose of this appendix is to derive the mean and variance of $f_Z(t)$.

The idea here is that we know the mean and variance of $f_1(z)$ and $f_2(z)$ which are chi-squared densities. The density of the r.v Z (2.23) is approximated by a convolution of $f_1(z)$ and $f_2(z)$, which is a density of a r.v Z' given by a sum of two independent r.v z_1 and z_2.

$$Z' = z_1 + z_2 \tag{C.1}$$

Consequently, the mean of Z' is the sum of the means and the variance of Z' is the sum of the variances. We also can find the mean and the variance of the original random variable Z as defined by (B.6). It is important to note that the mean and variance of Z will be found as a function of the system parameters. Once we have the mean and variance of Z' and Z we compare the appropriate terms and thus obtain the parameters σ_1^2 and σ_2^2 of $f_1(z)$ and $f_2(z)$. Since $f_1(z)$ and $f_2(z)$ are chi-squared densities, their mean and variances are given by

$$E(z_1) = n\sigma_1^2 \qquad \text{var}(z_1) = 2n\sigma_1^4 \tag{C.2}$$

$$E(z_2) = n\sigma_2^2 \qquad \text{var}(z_2) = 2n\sigma_2^4 \tag{C.3}$$

where we have noted that $f_2(z)$ is a chi-square on the negative axis.

Consequently we have for Z'

$$E\{Z'\} = E\{z_1 + z_2\} = n\sigma_1^2 + (-n\sigma_2^2)$$
$$= n(\sigma_1^2 - \sigma_2^2) \qquad (C.4)$$

$$\mathrm{var}(Z) = \mathrm{var}(z_1) + \mathrm{var}(z_2)$$
$$= 2n(\sigma_1^4 + \sigma_2^4)$$

We are now about to find the mean of Z via the likelihood-ratio.

It was shown (B.6)-(B.7) that the likelihood-ratio can be written in the following general form.

$$\Lambda_{k+1} = \exp\{\gamma_k + \frac{1}{2} x_k'(0)G_k(0)x_k(0) - \frac{1}{2}x_k'(1)G_k(1)x_k(1)\} \qquad (C.5)$$

Since γ_k is a constant we need only find the expected value of the following term

$$E\{Z\} = \frac{1}{2} E\{x_k'(0)G_k(0)x_k(0) - x_k'(1)G_k(1)x_k(1)\} \qquad (C.6)$$

Substituting into $x_k(0)$ and $x_k(1)$ their corresponding terms as defined by (B.5a) we obtain

$$E\{Z\} = \frac{1}{2} E\{(\tilde{x}_k'(0)A_k'C_{k+1}' + \sqrt{\theta}\ \omega_k'\Gamma'_k C'_{k+1} + v_{k+1}')(G_k(0))$$
$$\cdot (C_{k+1}A_k\tilde{x}_k(0) + \sqrt{\theta}\ C_{k+1}\Gamma_k\ \ \omega_k + v_{k+1})$$
$$- [(\tilde{x}_k'(1)A_k'C_{k+1}' + \sqrt{\theta}\ \omega_k'\Gamma'_k C'_{k+1} + v_{k+1}')(G_k(1))$$
$$\cdot (C_{k+1}A_k\tilde{x}_k(1) + \sqrt{\theta}\ C_{k+1}\Gamma_k\ \ \omega_k + v_{k+1})]\} \qquad (C.7)$$

Taking the expected value of (C.7) and after some algebra we find that $E\{Z\}$ is given by (C.8)

$$\beta_0 = E\{Z\} = \frac{1}{2}\{tr[P_{k|k}(0)A_k'C_{k+1}'G_k(0)C_{k+1}A_k]$$

$$+ tr[\theta Q_k\Gamma_k'C_{k+1}'G_k(0)C_{k+1}\Gamma_k] + tr[R_{k+1}G_k(0)]$$

$$- tr[P_{k|k}(1)A_k'C_{k+1}'G_k(1)C_{k+1}A_k] - tr[\theta Q_k\Gamma_k'C_{k+1}'G_k(1)C_{k+1}\Gamma_k]$$

$$- tr[R_{k+1}G_k(1)]\} \tag{C.8}$$

Proceeding to find the variance of Z consider the following term

$$\beta_1 = var\{Z\} = \frac{1}{4}E\{(x_k'(0)G_k(0)x_k(0) - x_k'(1)G_k(1)x_k(1))$$

$$\cdot (x_k'(0)G_k(0)x_k(0) - x_k'(1)G_k(1)x_k(1)^T\} - \beta_0^2$$

$$= \frac{1}{4}\{E\{x_k'(0)G_k(0)x_k(0)x_k'(0)G_k(0)x_k(0)\} - E\{x_k'(0)G_k(0)x_k(0)x_k'(1)G_k(1)x_k(1)\}$$

$$- E\{x_k'(1)G_k(1)x_k(1)x_k'(0)G_k(0)x_k(0)\}$$

$$+ E\{x_k'(1)G_k(1)x_k(1)x_k'(1)G_k(1)x_k(1)\} - \beta_0^2$$

$$= (1) + (2) + (3) \tag{C.9}$$

Since the evaluation of Equation (C.9) is very lengthy, each term will be evaluated individually.

(1) Taking the first term in (C.9), and expressing the double quadratic form with its indices. It was shown [18] that if x is Gaussian then a double quadratic form can be broken as follows

$$\sum_{i,j,k,\ell} G_{ij}(0)G_{k\ell}(0)[E\{x_i(0)x_j(0)\}E\{(x_k(0)x_\ell(0)\} + E\{x_i(0)x_k(0)\}E\{x_j(0)x_\ell(0)\}$$

$$+ E\{x_j(0)x_k(0)\}E\{x_i(0)x_\ell(0)\}]$$

$$= \sum_{i,j,k,\ell} G_{ij}(0)G_{k\ell}(0)\{P_{ij}(0)P_{k\ell}(0) + P_{ik}(0)P_{j\ell}(0) + P_{jk}(0)P_{i\ell}(0)\}$$

$$= \sum_{i,j,k,\ell} G_{ij}(0)G_{k\ell}(0)\{P_{ij}(0)P_{k\ell}(0) + 2P_{ik}(0)P_{j\ell}(0)\} \qquad (C.10)$$

performing the summation only on the first term of (C.10)

$$\sum_{i,j,k,\ell} G_{ij}(0)G_{k\ell}(0)P_{ij}(0)P_{k\ell}(0) = tr(G(0)P(0)tr(G(0)P(0)) \qquad (C.11)$$

consider now the second term of (C.10)

$$\sum_{i,j,k,\ell} 2G_{ij}(0)G_{k\ell}(0)P_{ik}(0)P_{j\ell}(0)$$

summing over the appropriate indices as follows

$$= [(G_{ij}P_{jk})G_{k\ell}]P_{i\ell}$$

$$= [(G(0)P(0))_{ik}G_{k\ell}]P_{i\ell}$$

$$= [(G(0)P(0)G(0))_{i\ell}]P_{i\ell}$$

$$= G(0)P(0)G(0)P'(0) => tr[G(0)P(0)G(0)P'(0)] \qquad (C.12)$$

(2)

$$= \sum_{i,j,k,\ell} G_{ij}(0)G_{k\ell}(1)[E(x_i(0)x_j(0))E(x_k(1)x_\ell(1)) + E(x_i(0)x_k(1))E(x_j(0)x_\ell(1))$$

$$+ E(x_j(0)x_k(1))E(x_i(0)x_\ell(1))]$$

$$= \sum_{i,j,k,\ell} G_{ij}(0)G_{k\ell}(1)[P_{ij}(0)P_{k\ell}(1) + 2S_{jk}S_{i\ell}] \qquad (C.13)$$

following the same procedure used in (C.10)-(C.12) we find that the solution to (C.13) is given by

$$= \mathrm{tr}[G(0)P(0)]\mathrm{tr}[G(1)P(1)] + 2\mathrm{tr}[G(0)SG(1)S'] \tag{C.14}$$

(3)

$$= \sum_{i,j,k,\ell} G_{ij}(1)G_{k\ell}(0)\{E(x_i(1)x_j(1))E(x_k(0)x_\ell(0)) + E(x_i(1)x_k(0))E(x_j(1)x_\ell(0))$$

$$+ E(x_j(1)x_k(0))E(x_i(1)x_\ell(0))\}$$

$$= \mathrm{tr}[G(1)P(1)]\mathrm{tr}[G(0)P(0)] + 2\mathrm{tr}[G(1)S'G(0)S] \tag{C.15}$$

After evaluating terms (1)-(3) we find that Eq. (C.9) becomes

$$\begin{aligned}\mathrm{var}\{Z\} &= \mathrm{tr}[G_k(0)P_k(0)]\mathrm{tr}[G_k(0)P_k(0)] + 2\mathrm{tr}[G_k(0)P_k(0)G_k(0)P_k'(0)] \\ &\quad - \mathrm{tr}[G_k(0)P_k(0)]\mathrm{tr}[G_k(1)P_k(1)] - 2\mathrm{tr}[G_k(0)S_kG_k(1)S_k'] \\ &\quad - \mathrm{tr}[G_k(1)P_k(1)]\mathrm{tr}[G_k(1)P_k(1)] - 2\mathrm{tr}[G_k(1)S_k'G_k(0)S_k] \\ &\quad + \mathrm{tr}[G_k(1)P_k(1)]\mathrm{tr}[G_k(1)P_k(1)] + 2\mathrm{tr}[G_k(1)P_k(1)G_k(1)P_k'(1)] - \beta_0^2 \\ &= \beta_1' - \beta_0^2\end{aligned} \tag{C.16}$$

The substitution of $P_k(i)$, $G_k(i)$, $i = 0,1$ and S_k into (C.16) result in the following

$$\mathrm{var}\{Z\} = E\{\{\tfrac{1}{2}(\tilde{x}_k'(0)A_k'C'_{k+1} + \sqrt{\theta}\ \omega_k'\Gamma_k'C_{k+1}' + v_{k+1}')\cdot G_k(0)$$

$$\cdot(C_{k+1}A_k\tilde{x}_k(0)+\sqrt{\theta}C_{k+1}\Gamma_k\omega_k+v_{k+1}) - \tfrac{1}{2}(\tilde{x}_k'(1)A_k'C_{k+1}'+\sqrt{\theta}\omega_k'\Gamma_k'C_{k+1}'+v_{k+1}')$$

$$\cdot G_k(1)\cdot(C_{k+1}A_k\tilde{x}_k(1) + \sqrt{\theta}\ C_{k+1}\Gamma_k\omega_k+v_{k+1})\}$$

$$\cdot\{\tfrac{1}{2}(\tilde{x}_k'(0)A_k'C_{k+1}' + \sqrt{\theta}\ \omega_k'\Gamma_k'C_{k+1}'+v_{k+1}')\cdot G_k(0)\cdot(\tilde{x}_k(0)A_kC_{k+1} + \sqrt{\theta}\ C_{k+1}\Gamma_k\omega_k$$

$$+ v_{k+1}) - \tfrac{1}{2}(\tilde{x}_k'(1)A_k'C_{k+1} + \sqrt{\theta}\ \omega_k'C_{k+1}+v_{k+1}')\cdot G_k(1)\cdot(C_{k+1}A_k\tilde{x}_k(1)$$

$$+ \sqrt{\theta}\ C_{k+1}\Gamma_k\omega_k+v_{k+1})\}^T\} - \beta_0^2 \tag{C.17}$$

After some extensive algebra we arrive to the final form of (C.12) given in (C.18)

$$\beta_1 = \beta_1' - \beta_0^2$$

$$+ \tfrac{1}{4}\ \mathrm{tr}(A_k'C_{k+1}'G_k(0)C_{k+1}A_kP_k(0))\mathrm{tr}(A_k'C_{k+1}'G_k(0)C_{k+1}A_kP_k(0))$$

$$+ \tfrac{\mathrm{tr}}{2}(A_k'C_{k+1}'G_k(0)C_{k+1}A_kP_k(0)A_k'C_{k+1}'G_k(0)C_{k+1}A_kP_k'(0))$$

$$+ \tfrac{1}{4}\ \mathrm{tr}(\theta\Gamma_k'C_{k+1}'G_k(0)C_{k+1}\Gamma_kQ_k)\mathrm{tr}(\theta\Gamma_k'C_{k+1}'G_k(0)C_{k+1}\Gamma_kQ_k)$$

$$+ \tfrac{\mathrm{tr}}{2}(\theta^2\Gamma_k'C'_{k+1}G_k(0)C_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}'G_k(0)C_{k+1}\Gamma_kQ_k')$$

$$+ \tfrac{1}{4}\ \mathrm{tr}(G_k(0)R_{k+1})\mathrm{tr}(G_k(0)R_{k+1}) + \tfrac{1}{2}\ \mathrm{tr}(G_k(0)R_{k+1}G_k(0)R_{k+1}')$$

$$- \tfrac{1}{2}\ \mathrm{tr}(A'_kC'_{k+1}G_k(0)C_{k+1}A_kP_k(0)\mathrm{tr}(A_k'C_{k+1}'G_k(1)C_{k+1}A_kP_k(1))$$

$$- \tfrac{1}{2}\ \mathrm{tr}(A_k'C_{k+1}'G_k(0)C_{k+1}A_kS_kA_k'C_{k+1}'G_k(1)C_{k+1}A_kS_k')$$

$$- \tfrac{1}{2}\ \mathrm{tr}(\theta\Gamma_kC_{k+1}'G_k(0)C_{k+1}\Gamma_kQ)\mathrm{tr}(\theta\Gamma_k'C_{k+1}'G_k(1)C_{k+1}\Gamma_kQ_k)$$

$$- \tfrac{1}{2}\ \mathrm{tr}(\theta^2\Gamma_k'C_{k+1}'G_kC_{k+1}\Gamma_kQ_k\Gamma_k'C_{k+1}'G_k(1)C_{k+1}\Gamma_kQ_k')$$

$$- \frac{1}{2}\,tr(G_k(0)R_{k+1})tr(G_k(1)R_{k+1}) - \frac{1}{2}\,tr(G_k(0)R_{k+1}G_k(1)R'_{k+1})$$

$$- \frac{1}{2}\,tr(A'_kC'_{k+1}G_k(1)C_{k+1}A_kS'_kA'_kC'_{k+1}G_k(0)C_{k+1}A_kS_k)$$

$$- \frac{1}{2}\,tr(\theta^2\Gamma'_kC'_{k+1}G_k(1)C_{k+1}\Gamma_kQ_k\Gamma'_kC'_{k+1}G_k(0)C_{k+1}\Gamma_kQ'_k) - \frac{1}{2}\,tr(G_k(1)R_{k+1}G_k(1)R'_{k+1})$$

$$- \frac{1}{4}\,tr(A'_kC'_{k+1}G_k(1)C_{k+1}A_kP_k(1))tr(A'_kC'_{k+1}G_k(1)C_{k+1}A_kP_k(1))$$

$$+ \frac{1}{2}\,tr(A'_kC'_{k+1}G_k(1)C_{k+1}A_kP_k(1)A'_kC'_{k+1}G_k(1)C_{k+1}A_kP'_k(1))$$

$$+ \frac{1}{4}\,tr(\theta\Gamma'_kC'_{k+1}G_k(1)C_{k+1}\Gamma_kQ_k)tr(\theta\Gamma'_kC'_{k+1}G_k(1)C_{k+1}\Gamma_kQ_k)$$

$$+ \frac{1}{2}\,tr(\theta^2\Gamma'_kC'_{k+1}G_k(1)C_{k+1}\Gamma_kQ_k\Gamma'_kC'_{k+1}G_k(1)C_{k+1}\Gamma_kQ'_k)$$

$$+ \frac{1}{4}\,tr(G_k(1)R_{k+1})tr(G_k(1)R_{k+1}) + \frac{1}{2}\,tr(G_k(1)R_{k+1}G_k(1)R'_{k+1}) - \beta_0^2 \qquad (C.18)$$

We now have two equations with two unknowns σ_1^2 and σ_2^2 .

$$n(\sigma_1^2 - \sigma_2^2) = \beta_0 \qquad (C.19)$$

$$2n(\sigma_1^2 + \sigma_2^2) = \beta_1 \qquad (C.20)$$

The solution of (C.19) and (C.20) for σ_1^2 and σ_2^2 yields

$$\sigma_1^2 = [(\frac{\beta_1}{4n}) - (\frac{\beta_0}{2n})^2]^{\frac{1}{2}} + (\frac{\beta_0}{2n}) \qquad (C.21)$$

$$\sigma_2^2 = [(\frac{\beta_1}{4n}) - (\frac{\beta_0}{2n})^2]^{\frac{1}{2}} - (\frac{\beta_0}{2n}) \qquad (C.22)$$

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